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Foreword

Math Mammoth Grade 7, International Version, comprises a complete maths curriculum for the seventh grade mathematics studies. This is a pre-algebra course, and students can continue to an Algebra 1 curriculum after completing it.

This curriculum is essentially the same as the U.S. version of Math Mammoth Grade 7, only customised for international audiences in a few aspects (listed below). The curriculum meets the Common Core Standards in the United States, but it may not perfectly align to the seventh grade standards in your country. However, you can probably find material for any missing topics in neighbouring grades.

The international version of Math Mammoth has been customised for international audiences in these aspects:

- The curriculum uses metric measurement units. Imperial units, such as miles and pounds, are not used, with the following exceptions. One exception is a problem about the heights of players in women's sports teams in Chapter 10, in the lesson "Comparing Two Populations". The statistical analysis can be done just by using the numbers; it is not important that they happen to be inches. Another exception is the usage of cups in several problems referring to serving sizes and recipes.
- The spelling conforms to British international standards.
- Paper size is A4.
- Large numbers are formatted with a space as the thousands separator (such as 12 394).
  (The decimals are formatted with a decimal point, the same as the US version.)

The main areas of study in Math Mammoth Grade 7 are:

- The basics of algebra (expressions, linear equations and inequalities);
- The four operations with integers and with rational numbers (including negative fractions and decimals);
- Ratios, proportions, and percent;
- Geometry;
- Probability and statistics.

This book, 7-B, covers ratios and proportions (chapter 6), percent (chapter 7), geometry (chapter 8), probability (chapter 9), and statistics (chapter 10). The rest of the topics are covered in the 7-A worktext.

I heartily recommend that you read the full user guide in the following pages.

I wish you success in teaching maths!

Maria Miller, the author

User Guide

Note: You can also find the information that follows online, at **https://www.mathmammoth.com/userguides/** .

Basic principles in using Math Mammoth complete curriculum

Math Mammoth is mastery-based, which means it concentrates on a few major topics at a time, in order to study them in depth. The two books (parts A and B) are like a “framework”, but you still have a lot of liberty in planning your student’s studies. You can even use it in a *spiral* manner, if you prefer. Simply have your student study in 2-3 chapters simultaneously.

In seventh grade, the first five chapters should be studied in order. Also, chapter 7 (Percent) should be studied before studying any of the chapters 8, 9, or 10 (Geometry, Probability, Statistics). You can be flexible with the rest of the scheduling.

Math Mammoth is not a scripted curriculum. In other words, it is not spelling out in exact detail what the teacher is to do or say. Instead, Math Mammoth gives you, the teacher, various tools for teaching:

- **The two student worktexts** (parts A and B) contain all the lesson material and exercises. They include the explanations of the concepts (the teaching part) in blue boxes. The worktexts also contain some advice for the teacher in the “Introduction” of each chapter.

  The teacher can read the teaching part of each lesson before the lesson, or read and study it together with the student in the lesson, or let the student read and study on his own. If you are a classroom teacher, you can copy the examples from the “blue teaching boxes” onto the board and go through them on the board.

- There are hundreds of **videos** matched to the curriculum available at https://www.mathmammoth.com/videos/ . There isn’t a video for every lesson, but there are dozens of videos for each grade level. You can simply have the author teach your student!

- Don’t automatically assign all the exercises. Use your judgement, trying to assign just enough for your student’s needs. You can use the skipped exercises later for revision. For most students, I recommend to start out by assigning about half of the available exercises. Adjust as necessary.

- For each chapter, there is a **link list to various free online games** and activities. These games can be used to supplement the maths lessons, for learning maths facts, or just for some fun. Each chapter introduction (in the student worktext) contains a link to the list corresponding to that chapter.

- The student books contain some **mixed revision lessons**, and the curriculum also provides you with additional **cumulative revision lessons**.

- There is a **chapter test** for each chapter of the curriculum, and a comprehensive end-of-year test.

- The **worksheet maker** allows you to make additional worksheets for most calculation-type topics in the curriculum. This is a single html file. You will need Internet access to be able to use it.

- You can use the free online exercises at https://www.mathmammoth.com/practice/
  This is an expanding section of the site, so check often to see what new topics we are adding to it!

- Some grade levels have **cut-outs** to make fraction manipulatives or geometric solids.

- The answer keys are included in the digital download version. They are sold as a separate book for the printed version.

How to get started

Have ready the first lesson from the student worktext. Go over the first teaching part (within the blue boxes) together with your student. Go through a few of the first exercises together, and then assign some problems for your student to do on their own.

Repeat this if the lesson has other blue teaching boxes. Naturally, you can also use the videos at https://www.mathmammoth.com/videos/

Many students can eventually study the lessons completely on their own — the curriculum becomes self-teaching. However, students definitely will vary in how much they need someone to be there to actually teach them.

Pacing the curriculum

Each chapter introduction contains a suggested pacing guide for that chapter. You will see a summary on the right. (This summary does not include the optional lessons nor tests.)

Most lessons are 3 or 5 pages long, intended for one day. Some 5-page lessons can take two days. There are also a few optional lessons.

It can also be helpful to calculate a general guideline as to how many pages per week the student should cover in order to go through the curriculum in one school year.

Worktext 7-A	
Chapter 1	8 days
Chapter 2	13 days
Chapter 3	9 days
Chapter 4	16 days
Chapter 5	16 days
TOTAL	**62 days**

Worktext 7-B	
Chapter 6	16 days
Chapter 7	11 days
Chapter 8	22 days
Chapter 9	10 days
Chapter 10	12 days
TOTAL	**71 days**

The table below lists how many pages there are for the student to finish in this particular grade level, and gives you a guideline for how many pages per day to finish, assuming a 180-day (36-week) school year. The page count in the table below *includes* the optional lessons.

Example:

Grade level	School days	Days for tests and revisions	Lesson pages	Days for the student book	Pages to study per day	Pages to study per week
7-A	83	10	199	73	2.73	13.6
7-B	97	10	240	87	2.76	13.8
Grade 7 total	180	20	439	160	2.74	13.7

The table below is for you to use.

Grade level	School days	Days for tests and revisions	Lesson pages	Days for the student book	Pages to study per day	Pages to study per week
7-A			199			
7-B			240			
Grade 7 total			439			

Let’s say you determine that your student needs to study about 2.7 pages a day, or 14 pages a week on average in order to finish the curriculum in a year. As the student studies each lesson, keep in mind that sometimes most of the page might be reserved for workspace to solve equations or other problems. You might be able to cover more than the average number of pages on such a day. Some other day you might assign the student only one page of word problems.

Now, something important. Whenever the curriculum has lots of similar practice problems (a large set of problems), feel free to **only assign 1/2 or 2/3 of those problems**. If your student gets it with less amount of exercises, then that is perfect! If not, you can always assign the rest of the problems for some other day. In fact, you could even use these unassigned problems the next week or next month for some additional revision.

In general, seventh graders might spend 45-90 minutes a day on maths. If your student finds maths enjoyable, they can of course spend more time with it! However, it is not good to drag out the lessons on a regular basis, because that can affect the student's attitude towards maths.

Working space, the usage of additional paper and mental maths

The curriculum generally includes working space directly on the page for students to work out the problems. However, feel free to let your students to use extra paper when necessary. They can use it, not only for the "long" algorithms (where you line up numbers to add, subtract, multiply, and divide), but also to draw diagrams and pictures to help organise their thoughts. Some students won't need the additional space (and may resist the thought of extra paper), while some will benefit from it. Use your discretion.

Some exercises don't have any working space, but just an empty line for the answer (e.g. 200 + ______ = 1000). Typically, I have intended that such exercises be done using MENTAL MATHS.

However, there are some students who struggle with mental maths (often this is because of not having studied and used it in the past). As always, the teacher has the final say (not me!) as to how to approach the exercises and how to use the curriculum. We do want to prevent extreme frustration (to the point of tears). The goal is always to provide SOME challenge, but not too much, and to let students experience success enough so that they can continue enjoying learning maths.

Students struggling with mental maths will probably benefit from studying the basic principles of mental calculations from the earlier levels of Math Mammoth curriculum. To do so, look for lessons that list mental maths strategies. They are taught in the chapters about addition, subtraction, place value, multiplication, and division. My article at https://www.mathmammoth.com/lessons/practical_tips_mental_math also gives you a summary of some of those principles.

Using tests

For each chapter, there is a **chapter test**, which can be administered right after studying the chapter. **The tests are optional.** Some families might prefer not to give tests at all. The main reason for the tests is for diagnostic purposes, and for record keeping. These tests are not aligned or matched to any standards.

The end-of-year test is best administered as a diagnostic or assessment test, which will tell you how well the student remembers and has mastered the mathematics content of the entire grade level.

Using cumulative revisions and the worksheet maker

The student books contain mixed revision lessons which revise concepts from earlier chapters. The curriculum also comes with additional cumulative revision lessons, which are just like the mixed revision lessons in the student books, with a mix of problems covering various topics. These are found in their own folder in the digital version, and in the Tests & Cumulative Revisions book in the printed version.

The cumulative revisions are optional; use them as needed. They are named indicating which chapters of the main curriculum the problems in the revision come from. For example, "Cumulative Revision, Chapter 4" includes problems that cover topics from chapters 1-4.

Both the mixed and cumulative revisions allow you to spot areas that the student has not grasped well or has forgotten. When you find such a topic or concept, you have several options:

1. Check if the worksheet maker lets you make worksheets for that topic.
2. Check for any online games and resources in the Introduction part of the particular chapter in which this topic or concept was taught.
3. If you have the digital version, you could simply reprint the lesson from the student worktext, and have the student restudy that.
4. Perhaps you only assigned 1/2 or 2/3 of the exercise sets in the student book at first, and can now use the remaining exercises.
5. Check if our online practice area at https://www.mathmammoth.com/practice/ has something for that topic.
6. Khan Academy has free online exercises, articles, and videos for most any maths topic imaginable.

Concerning challenging word problems and puzzles

While this is not absolutely necessary, I heartily recommend supplementing Math Mammoth with challenging word problems and puzzles. You could do that once a month, for example, or more often if the student enjoys it.

The goal of challenging story problems and puzzles is to **develop the student's logical and abstract thinking and mental discipline**. I recommend starting these in fourth grade, at the latest. By then, students are able to read the problems on their own and have developed mathematical knowledge in many different areas. Of course I am not discouraging students from doing such in earlier grades, either.

Math Mammoth curriculum contains lots of word problems, and they are usually multi-step problems. Several of the lessons utilise a bar model for solving problems. Even so, the problems I have created are usually tied to a specific concept or concepts. I feel students can benefit from solving problems and puzzles that require them to think "outside the box" or are just different from the ones I have written.

I recommend you use the free Math Stars problem-solving newsletters as one of the main resources for puzzles and challenging problems:

Math Stars Problem Solving Newsletter (grades 1-8)
https://www.homeschoolmath.net/teaching/math-stars.php

I have also compiled a list of other resources for problem solving practice, which you can access at this link:

https://l.mathmammoth.com/challengingproblems

Another idea: you can find puzzles online by searching for "brain puzzles for kids," "logic puzzles for kids" or "brain teasers for kids."

Frequently asked questions and contacting us

If you have more questions, please first check the FAQ at https://www.mathmammoth.com/faq-lightblue

If the FAQ does not cover your question, you can then contact us using the contact form at the MathMammoth.com website.

Chapter 6: Ratios and Proportions
Introduction

Chapter 6 revises the concept, which has already been presented in grade 6, of the **ratio** of two quantities. From this concept, we develop the related concepts of a **rate** (so much of one thing per so much of another thing) and a **proportion** (an equation of two ratios).

When two quantities are in proportion, we can consider the quantities as variables, write an equation to describe the relationship between them, and graph that equation. This study of proportional relationships takes the concept of *ratio* to a new level, and paves the way to the study of linear functions in 8th grade.

The first lessons focus on the concepts of ratio, rate, and unit rate. Students use tables of equivalent ratios and unit rates to solve a variety of problems involving rates. We especially focus on calculating unit rates when the quantities involve fractions.

Then we study proportional relationships, using the familiar tables of equivalent rates as a starting point. Students write and graph equations relating the two quantities (seen as variables now). They find the unit rate and plot it on the graph as a single point, and relate the different representations of proportional relationships (graph, table of values, wording, and equation) to each other. We also spend some time analysing whether a given relationship between variables is proportional or not.

The next topic is proportions — equations where one ratio is equal to another. Students learn to solve proportions with cross-multiplying and to set them up in the correct way to solve a word problem. They also learn and compare different ways to solve problems with rates. It is not always necessary to set up a proportion!

Then we turn our attention to an application of all this in geometry: scaled figures, scale drawings, floor plans, and maps. Students encounter scales such as 1:90 or 2 cm = 30 m. They calculate dimensions in reality from the scale drawing and vice versa, and redraw scale drawings at a different scale. Floor plans use a scale also, and are hopefully an interesting topic to students.

The lesson on maps is optional. In today's world, most of us are using online map services which calculate the distances for us, so there is much less need to figure out distances using physical maps, but some students (and teachers) might find the topic interesting.

In 8th grade, students encounter proportional relationships again, as they learn the connection between the unit rate and the slope of the graph, and compare different proportional relationships represented in different ways.

I recommend not assigning all the exercises by default, but that you use your judgement, and strive to vary the number of assigned exercises according to the student's needs. Please see the user guide at **https://www.mathmammoth.com/userguides/** for more guidance on using and pacing the curriculum.

There are free videos matched to the curriculum at **https://www.mathmammoth.com/videos/** (choose 7th grade).

Good Mathematical Practices

- The chapter gives students many opportunities to persevere in problem solving through various multi-step word problems. Remind your student(s) that making a mistake is part of normal problem solving, and when you think about your mistake, your brain actually grows. There is no brain growth when doing problems that you already know how to solve.
- The study of proportional relationships lays a big foundational piece for mathematical modelling. In this chapter, students encounter many real-life situations (e.g. speed, price per kg). They learn to see the quantities involved as variables, to write an equation relating the variables, and to graph the equations. These are all essential skills in mathematical modelling.
- In the lesson Scale Drawings 2, students will explore how the area of a scaled figure relates to the scale factor, and they have the opportunity to find the general rule through repeated reasoning.

Pacing Suggestion for Chapter 6

This table does not include the chapter test because it is found in a different book (or file).
Please add one day to the pacing for the test if you use it.

The Lessons in Chapter 6	**page**	**span**	**suggested pacing**	**your pacing**
Ratios and Rates	14	*3 pages*	1 day	
Solving Problems Using Equivalent Rates	17	*3 pages*	1 day	
Unit Rates	20	*4 pages*	1 day	
Proportional Relationships 1	24	*4 pages*	1 day	
Proportional Relationships 2	28	*4 pages*	1 day	
Proportional Relationship or Not?	32	*4 pages*	1 day	
Solving Proportions	36	*3 pages*	1 day	
Proportions and Problem Solving	39	*4 pages*	1 day	
More on Proportions	43	*4 pages*	1 day	
Scaling Figures	47	*4 pages*	1 day	
Scale Drawings 1	51	*3 pages*	1 day	
Floor Plans	54	*3 pages*	1 day	
Scale Drawings 2	57	*3 pages*	1 day	
Scale Drawings—More Practice (optional)	60	*2 pages*	1 day	
Maps (optional)	62	*4 pages*	2 days	
Chapter 6 Mixed Revision	66	*3 pages*	1 day	
Chapter 6 Revision	69	*5 pages*	2 days	
Chapter 6 Test (optional)				
TOTALS		*54 pages*	16 days	
with optional content		*(60 pages)*	(19 days)	

Helpful Resources on the Internet

We have compiled a list of Internet resources that match the topics in this chapter, including pages that offer:

- **online practice** for concepts;
- online **games**, or occasionally, printable games;
- **animations** and interactive **illustrations** of maths concepts;
- **articles** that teach a maths concept.

We heartily recommend you take a look! Many of our customers love using these resources to supplement the bookwork. You can use these resources as you see fit for extra practice, to illustrate a concept better and even just for some fun. Enjoy!

https://l.mathmammoth.com/gr7ch6

SCAN ME

Ratios and Rates

A **ratio** is a comparison of two numbers, or quantities, using division.

For example, to compare the hearts to the stars in the picture, we say that the ratio of hearts to stars is 5:10 (read “five to ten”).

The two numbers in the ratio are called the **first term** and the **second term** of the ratio. The order in which these terms are mentioned does matter! For example, the ratio of stars to hearts is *not* the same as the ratio of hearts to stars. The former is 10:5 and the latter is 5:10.

We can write this ratio in several different ways:

- The ratio of hearts to stars is 5:10.
- The ratio of hearts to stars is 5 to 10.
- The ratio of hearts to stars is $\frac{5}{10}$.
- For every five hearts, there are ten stars.

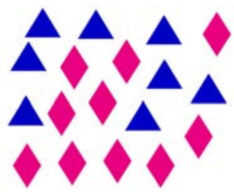
Note that we are not comparing two numbers to determine which one is greater (as in 5 < 10). The comparison is relative as in a multiplication problem. For example, the ratio 5:10 can be simplified to 1:2, and it indicates to us that there are twice as many stars as there are hearts.

We **simplify ratios** in exactly the same way we simplify fractions.

Example 1. In the picture at the right, the ratio of hearts to stars is 12:16. We can simplify that ratio to 6:8 and even further to 3:4. These three ratios (12:16, 6:8, and 3:4) are called **equivalent ratios**.

The ratio that is simplified to lowest terms, 3:4, tells us that for every three hearts, there are four stars.

1. Write the ratio and then simplify it to lowest terms.



The ratio of triangles to diamonds is _____ : _____ = _____ : _____ .

In this picture, there are ______ triangles to every _______ diamonds.

2. **a.** Draw a picture with pentagons and circles so that the ratio of pentagons to the total of all the shapes is 7:9.

   b. What is the ratio of circles to pentagons?

3. **a.** Draw a picture in which (1) there are three diamonds for every five triangles, and (2) there is a total of 9 diamonds.

   b. Write the ratio of all the diamonds to all the triangles, and simplify this ratio to lowest terms.

4. Write the equivalent ratios.

a. 5 to 45 = 1 to ________	**b.** 3 : _____ = 9 : 60	**c.** 280 : 420 = 2 : _____	**d.** $\frac{5}{13} = \frac{\square}{65}$

We can also form **ratios using quantities that have units**. If the units are the same, they cancel.

Example 2. Simplify the ratio 250 g : 1.5 kg.

First we convert 1.5 kg to grams and then simplify: $\frac{250\text{ g}}{1.5\text{ kg}} = \frac{250\text{ g}}{1500\text{ g}} = \frac{250}{1500} = \frac{1}{6}$.

5. Use a fraction line to write ratios of the given quantities as in the example. Then simplify the ratios.

a. 5 kg and 800 g $\frac{5\text{ kg}}{800\text{ g}} =$	**b.** 600 cm and 2.4 m
c. 1 litre and 750 ml	**d.** 325 cm and 1.25 m

We can generally **convert** ratios with decimals or fractions **into ratios of whole numbers**.

Example 3. Because we can multiply both terms of the ratio by 10, $\frac{1.5\text{ km}}{2\text{ km}} = \frac{15\text{ km}}{20\text{ km}}$.

Then: $\frac{15\text{ km}}{20\text{ km}} = \frac{15}{20} = \frac{3}{4}$. So the ratio 1.5 km : 2 km is equal to 3:4.

You can also see that the ratio is 3:4 by noticing that both 1.5 km and 2 km are evenly divisible by 500 m.

Example 4. Simplify the ratio ¼ mile to 5 miles.

First, the units cancel: ¼ mi : 5 mi = ¼ : 5. Multiplying both terms of the ratio by 4, we get ¼ : 5 = 1:20.

6. Use a fraction line to write ratios of the given quantities. Then simplify the ratios to whole numbers.

a. 5.6 km and 3.2 km	**b.** 0.02 m and 0.5 m
c. 1.25 m and 0.5 m	**d.** 1/2 L and 7 1/2 L
e. 1/4 cup and 3 1/2 cups	**f.** 2/3 km and 1 km

If the two terms in a ratio have *different* units, then the ratio is also called a **rate.**
Example 5. The ratio "8 km to 40 minutes" is a rate that compares the quantities "8 km" and "40 minutes," perhaps for the purpose of giving us the speed at which a person is running.   We can write this rate as 8 km : 40 minutes or $\frac{\text{8 km}}{\text{40 minutes}}$ or 8 km *per* 40 minutes.   The word "per" in a rate signifies the same thing as a colon or a fraction line.
This rate can be simplified: $\frac{\text{8 km}}{\text{40 minutes}} = \frac{\text{1 km}}{\text{5 minutes}}$. The person runs 1 km in 5 minutes.
Example 6. Simplify the rate "15 pencils per 100¢." Solution: $\frac{\text{15 pencils}}{\text{100¢}} = \frac{\text{3 pencils}}{\text{20¢}}$.

7\. Write each rate using a colon, the word "per," or a fraction line. Then simplify it.

a. Jeff swims at a constant speed of 400 metres : 15 minutes.

b. The car can travel 45 km on 3 L of petrol.

8\. Fill in the missing terms to form equivalent rates.

a. $\frac{\text{1/2 cm}}{\text{30 min}} = \frac{\quad}{\text{1 h}} = \frac{\quad}{\text{15 min}}$	**b.** $\frac{\$88.40}{\text{8 hr}} = \frac{\quad}{\text{2 hr}} = \frac{\quad}{\text{10 hr}}$

9\. Simplify these rates. Don't forget to write the units.

a. 280 km per 7 hours	**b.** 10 cm : 1.5 minutes

10\. A car is travelling at a constant speed of 72 km/hour. Fill in the table of equivalent rates: each pair of numbers in the table (distance/time) forms a rate that is equivalent to the rate 72 km/hour.

Distance (km)							
Time (min)	10	30	40	50	60	90	100

11\. Eight pairs of socks cost $20. Fill in the table of equivalent rates.

Cost ($)								
Pairs of socks	1	2	4	6	7	8	9	10

Solving Problems Using Equivalent Rates

Example 1. It took Liam 1 ½ hours to paint 8 metres of fence. Painting at the same speed, how long will it take him to paint the rest of the fence, which is 28 metres long?

In this problem, we see a rate of 8 m per 1 ½ hours. There is another rate, too: 28 m per an unknown amount of time. These two are equivalent rates. We can use a table of equivalent rates to solve the problem.

Amount of fence (m)	8	4	28
Time (minutes)	90	45	315

(1) We figure that Liam can paint 4 m of fence in 45 minutes (by dividing the terms in the original rate by 2).

(2) Next we multiply both terms in the rate of 4 m/45 min by seven to get the rate 28 m/315 min.

It will take Liam 315 minutes, or 5 hours 15 minutes, to paint the rest of the fence.

Example 2. Sofia rides her bike 20 km in 45 minutes. Riding at the same speed, how far will she go in 1 hour?

We can multiply or divide both terms of a rate by the same number to form another, equivalent rate. (You have used this same idea in the past with equivalent fractions.)

$$\frac{20 \text{ km}}{45 \text{ minutes}} \overset{\div 3}{\underset{\div 3}{=}} \frac{\boxed{} \text{ km}}{15 \text{ minutes}} \overset{\cdot 4}{\underset{\cdot 4}{=}} \frac{\boxed{} \text{ km}}{60 \text{ minutes}}$$

It's not easy to go directly from 45 minutes to 60, but we can use 15 as a "stepping stone" in between.

Recall that 20 ÷ 3 is easy to solve when you think of it as a fraction: 20/3 = 6 2/3. Sofia can ride 6 2/3 km in 15 minutes.

Then, we multiply both terms of that rate by 4. Again, don't be intimidated by the fraction: 4 · (6 2/3) = 4 · (20/3) = 80/3 = 26 2/3. So, Sofia can ride 26 2/3 km in 1 hour.

1. Fill in the tables of equivalent rates.

a.

Distance	15 km			
Time	3 hr	1 hr	15 min	45 min

b.

Pay	$15			
Time	45 min	15 min	1 hr	1 hr 45 min

2. Fill in the missing terms in these equivalent rates.

a. $\dfrac{3 \text{ pies}}{8 \text{ boys}} = \dfrac{}{2 \text{ boys}} = \dfrac{}{12 \text{ boys}} = \dfrac{}{20 \text{ boys}}$

b. $\dfrac{115 \text{ words}}{2 \text{ min}} = \dfrac{}{1 \text{ min}} = \dfrac{}{3 \text{ min}}$

3. Aiden can ride his electric bicycle 12 km in 28 minutes. At the same constant speed, how long will he take to go 54 km?

$$\frac{12 \text{ km}}{28 \text{ minutes}} = \frac{6 \text{ km}}{\boxed{} \text{ minutes}} = \frac{\boxed{} \text{ km}}{\boxed{} \text{ minutes}}$$

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Revision: Percent

Percent (or **per cent**) means *per hundred* or "divided by a hundred." (The word "cent" means one hundred.) So, percent means the rate per hundred, or a hundredth part.

To convert percentages into fractions, simply read the "per cent" as "per 100." Thinking of hundredths, you can also easily write them as decimals.

Therefore, 8% = 8 per cent = 8 per 100 = 8/100 = 0.08.

Similarly, 167% = 167 per 100 = 167/100 = 1.67.

$\frac{5}{100}$ five per cent = 5%

1. Write as percentages, fractions, and decimals.

Percent	Decimal	Fraction
	0.07	
52%		
		$\frac{59}{100}$

Percent	Decimal	Fraction
109%		
200%		
		$\frac{382}{100}$

A number with two decimal digits has hundredths, so it can easily be written as a percentage. For example, 0.56 = 56%. But we can write numbers with more decimal digits as percents, also.

Example 1. As a percentage, the number 0.5642 is 56.42%. Compare this to 0.56 = 56%. The digits "42" simply follow the digits "56", and become the decimal digits for the percentage.

Decimal	Percent	Fraction
0.09	9%	$\frac{9}{100}$
0.091	9.1%	$\frac{91}{1000}$
0.09146	9.146%	$\frac{9146}{100\,000}$

2. Write as percentages, fractions, and decimals.

Percent	Decimal	Fraction
0.9%		
		$\frac{282}{1000}$
	0.8914	

Percent	Decimal	Fraction
		$\frac{91}{10\,000}$
2.391%		
	0.94284	

Writing fractions as percentages

Example 2. Sometimes you can convert a fraction into an equivalent fraction with a denominator of 100, 1000, or some other power of 10. After that it is easy to write it as a decimal and then as a percentage.

$$\frac{46}{25} \overset{\cdot 4}{=} \frac{184}{100} = 1.84 = 184\%$$

Example 3. For most fractions, we need to use *division* to convert the fraction to a decimal first, and then to a percentage.

Simply treat the fraction line as a division symbol and divide (using long division or a calculator), to get a decimal. Then write it as a percentage.

$$\frac{8}{9} = 0.888... \approx 0.889 = 88.9\%$$

$$\begin{array}{r} 0.8888 \\ 9\overline{)8.0000} \\ \underline{-7\,2} \\ 8\,0 \\ \underline{-7\,2} \\ 8\,0 \\ \underline{-7\,2} \\ 8\,0 \\ \underline{-7\,2} \\ 8 \end{array}$$

3. Fill in the table. First write each fraction as an equivalent fraction where the denominator is a power of ten.

Fraction	Fraction (denominator is a power of ten)	Decimal	Percent
$\frac{8}{25}$	$\frac{\square}{100}$		
$\frac{142}{200}$	$\frac{\square}{100}$		
$\frac{24}{20}$			
$\frac{31}{250}$			
$\frac{3}{8}$			

4. Write as percentages. Use long division. Round your answers to the nearest tenth of a percent.

a. 11/8

b. 11/24

5. Write the fractions as decimals and percentages. Round the decimals to four decimal digits. Use a calculator.

a. $\frac{2}{3} \approx$ ____________ = ____________%	**b.** $\frac{11}{6} \approx$ ____________ = ____________%
c. $\frac{17}{23} \approx$ ____________ = ____________%	**d.** $\frac{304}{57} \approx$ ____________ = ____________%

6. Match the fractions and percentages.

$\frac{3}{4}$	$\frac{17}{20}$	$\frac{4}{5}$	$\frac{18}{25}$	$\frac{5}{4}$	$\frac{7}{10}$	$\frac{9}{8}$	$\frac{23}{20}$
125%	80%	75%	115%	70%	72%	85%	112.5%

7. Remember mental maths? Fill in the shortcuts for finding these easy percentages of a number.

To find 50% of a number, divide it by _______. To find 25% of a number, divide it by _______.	To find 10% of a number, divide it by ________. To find 1% of a number, divide it by ________.
To find 30% of a number, first find __________% of the number, then multiply that by _____. To find 75% of a number, first find __________% of the number, then multiply that by _____.	

8. Find various percentages of the number 360.

Percentage	Value
5%	
10%	
20%	
25%	
50%	
60%	
75%	
80%	
100%	360
125%	
150%	

9. Solve using mental maths.

a. What is 25% of $84.00?

b. 17 is 25% of what number?

c. Find 60% of 300.

d. 8 is 1% of what number?

e. Find 3% of 2000 km.

f. 9 is 3% of what number?

g. What is 20% of $45?

h. 24 is 60% of what number?

i. Find 150% of $60.

j. 36 is 200% of what number?

Solving Basic Percentage Problems

All percentages are fractions. Recall that “percent” means “per 100”. For example, 34% is 34 per 100 or 34/100 — a fraction. Stated differently, a percentage is a “rate per 100”.

All percentage problems have to do with a **part** versus **total**. As a fraction, we write $\frac{part}{total}$.

As a percent, it is still a fraction, and has the same value, but we want the denominator to be 100.

For example, to find what percentage 2 is of 5, we can write: $\frac{2}{5} = \frac{40}{100}$, and then write 40/100 as 40%.

Example 1. What percentage is 14 km of 75 km?

We know the total (75 km) and we know the part (14 km). This is essentially asking what fraction 14 km is of 75 km, but we need to express the answer as a percentage, not as a fraction. Therefore:

1. We write the fraction $\frac{part}{total}$: it is $\frac{14\text{ km}}{75\text{ km}}$ but the units “km” cancel out so it becomes just $\frac{14}{75}$.

2. Then we use a calculator to divide $14/75 = 0.18\overline{6}$ and write that as a percentage: $0.18\overline{6} = 18.\overline{6}\%$.

Normally, we round the result and say that 14 km is about 19% of 75 km.

Example 2. Find 59.2% of $2600.

Here we know the percentage — which means we know the fraction — and the total. We don’t know the *part* as a quantity. This is the same as asking for 592/1000 of $2600.

Recall that the word “of” translates into multiplication. Thus, we could use fraction multiplication (we could calculate (592/1000) · 2600) but often, the quickest way to do these types of calculations is to convert the percentage into a decimal first, and then use <u>decimal</u> multiplication.

So, instead of (592/1000) · $2600, we write 59.2% as 0.592, and calculate 0.592 · $2600 = $1539.20.

If the percentage is known and the total is known: (What is x% of y?)

This is the same as asking for a fraction of some total.

1. Write the percentage as a decimal.
2. Multiply that decimal by the total.

Or use mental maths tricks for finding 1%, 10%, 20%, 30%, 25%, 50%, 75%, *etc*. of a number.

If you are asked the percentage:

Asking “what percentage” is essentially the same as asking “what part” or “what fraction.”

1. Write the fraction $\frac{part}{total}$.

2. Write this fraction as a percentage. Often, you will do this in two steps: first write that fraction as a decimal, and then that as a percentage.

1\. Fill in the tables. Use mental maths.

a.

Amount			75					
Percentage	10%	20%	30%	50%	100%	120%	150%	200%

b.

Amount		12	20	32			60	
Percentage	10%				100%	120%	150%	200%

You may use a calculator for the rest of the problems in the lesson except question #8.

2. A shirt that cost $34 was discounted by $4. What is the discount percent?

3. Julia paid $325.08 of her $1890 pay cheque in taxes.
   What percentage of her pay cheque did she pay in taxes?

4. At 7 years of age, Matthew was at 68.7% of his adult height, which is 182 cm.
   How tall was Matthew when he was 7?

5. On a certain Monday, 5.3% of a school's 980 students didn't show up.
   How many students were at school that day?

6. Harry has two roosters, named Captain and Chief. The weight of Captain is 7/5 of the weight of Chief.

   a. Write the second sentence above using a percentage instead of a fraction.

   b. If Chief weighs 3 kg, how much does Captain weigh?

7. The Carters live on a rectangular piece of land that measures 40 m × 35 m. The Joneses live on a rectangular piece of land that measures 42 m × 39 m.

   a. To the nearest hundredth of a percent, find what percentage the area of the Carters' land is of the area of the Joneses' land.

   b. If the Joneses' land is valued at $4914, and the Carters' land is appraised at the same value per square metre, what would be the value of the Carters' land?

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Angle Relationships 1

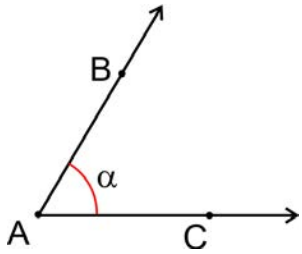
A **ray** has a starting point and continues indefinitely in one direction (indicated by one arrowhead).

In contrast to a ray, a **line** continues indefinitely in *two* directions (indicated by two arrowheads).

An **angle** consists of **two rays that start at the same point**, called the **vertex**. Each ray is called a **side** of the angle.

We can denote the angle on the right as angle BAC, or using the symbol "∠" for "angle," as ∠BAC.

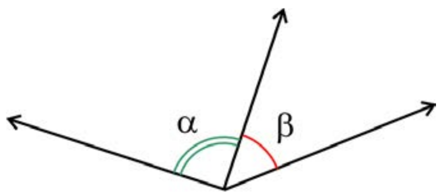
Note that we list the vertex point in the middle: it is ∠B**A**C, not ∠ABC. We could also name it ∠CAB.



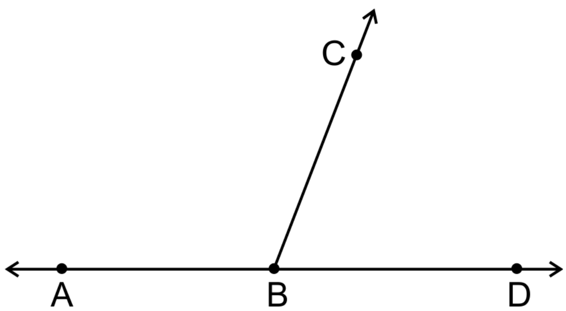
In mathematics, we also often denote angles with the beginning letters of the Greek alphabet: α (alpha), β (beta), γ (gamma), and δ (delta). So ∠BAC can also be called "angle α."

Two angles are **adjacent** if they have a **common vertex and share one side.**

In the image on the right, ∠α and ∠β are adjacent (side-by-side) angles.



1. B is a point on line AD. Find the measures of the three angles, and also the angle sums. Do you notice any special numbers?



∠ABC = __________°

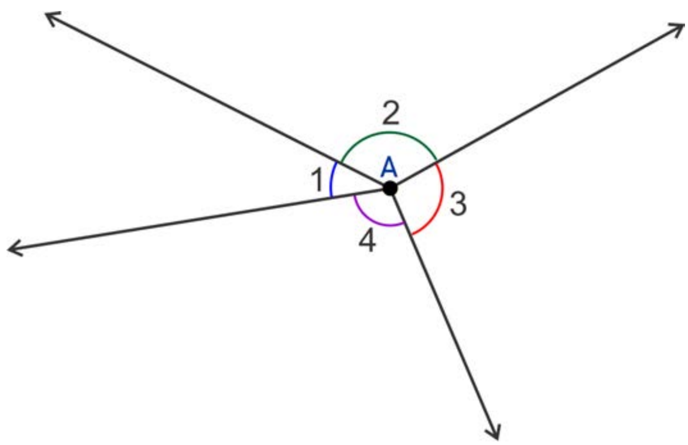
∠CBD = __________°

∠ABD = __________°

sum of ∠ABC and ∠CBD :   ____________°

sum of all three angles: _____________ °   (This should be 360°.)

2. Several rays start at point A. Measure the angles. Calculate their sum.

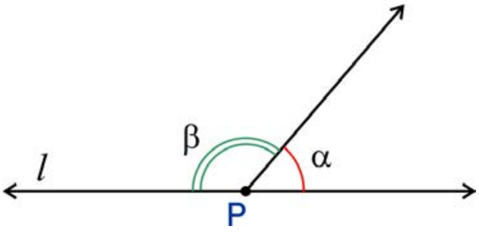


∠1 = __________°   ∠2 = __________°

∠5 = __________°   ∠4 = __________°

Sum of the angles = __________°

P is a point on line *l*. The angles ∠α and ∠β in this image are adjacent, and they form a straight angle (an angle of 180 degrees). They are called **supplementary angles**.

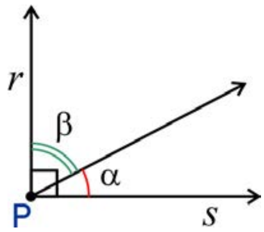


Two angles are supplementary if their **sum is 180 degrees:**

$$\angle\alpha + \angle\beta = 180°$$

We also say that α supplements β.

Rays *r* and *s* start at point P and form a right angle. The adjacent angles ∠α and ∠β form a right angle. They are called **complementary angles**.



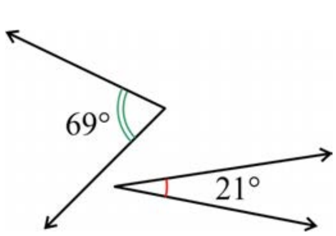
Two angles are complementary if their **sum is 90 degrees:**

$$\angle\alpha + \angle\beta = 90°$$

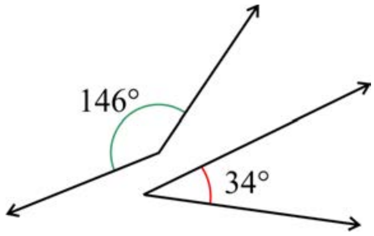
We also say that α complements β.

Here's a mnemonic to help you remember the difference: **S**upplementary angles form a **S**traight line, and **C**omplementary angles form a **C**orner (a right angle).

Supplementary angles don't have to be adjacent, and neither do complementary angles.



These are still complementary angles, because 21° + 69° = 90°.



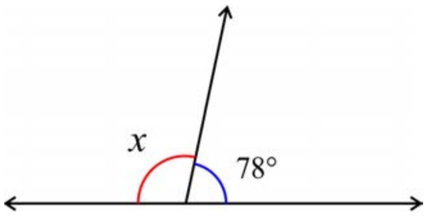
These are still supplementary angles, because 146° + 34° = 180°.

3\. **a.** Draw a 38° angle. Then draw an adjacent angle that complements it.

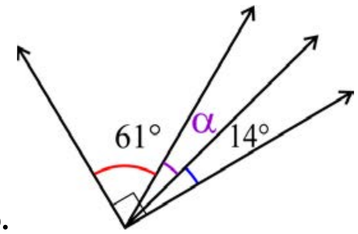
b. Draw an 82° angle. Then draw an adjacent angle that supplements it.

4. Write an equation for the unknown and solve it. Do not measure any angles.

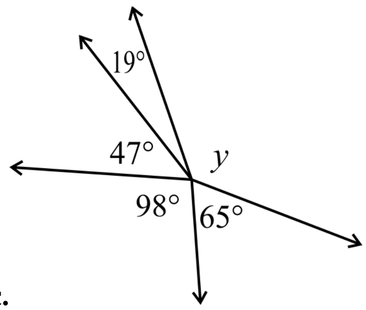
a.



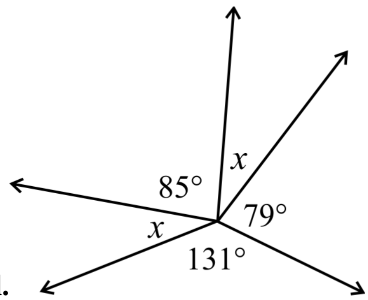
b.



c.

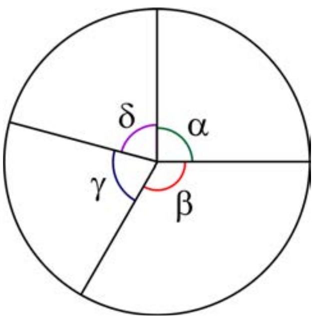


d.



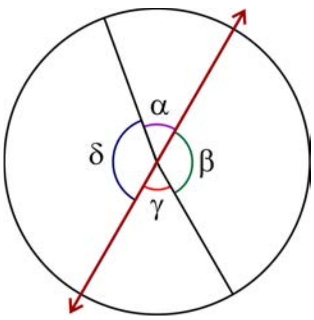
5. Figure out the missing entries in the tables without measuring any angles.

a.



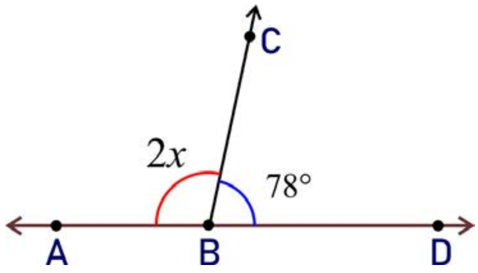
Angle	Degrees	Fraction	Percentage
α		1/4	
β	120°		
γ			
δ	75°		

b.



Angle	Degrees	Fraction	Percentage
α	50°		
β			
γ		1/6	
δ			

Example. Point B is on line AD. Write an equation to solve for the unknown. What is the measure of angle ABC?



Since angle ABD is a straight angle (180°), the equation is:

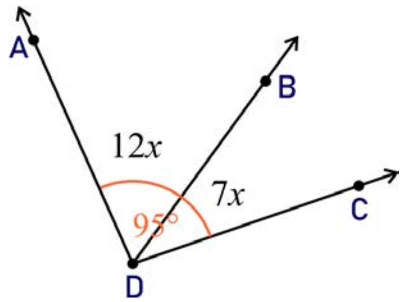
$$\begin{aligned} 2x + 78 &= 180 \\ 2x &= 102 \\ x &= 51 \end{aligned}$$

So, x is 51°. However, angle ABC does not measure 51° because its measure is $2x$, not x. So, we double the value of x to get that ∠ABC = 102°.

6. Angle ADC measures 95°.

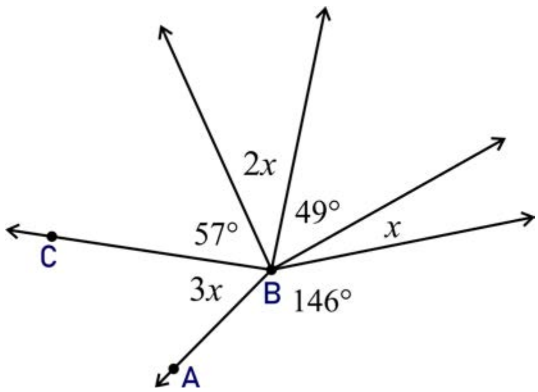
a. Write an equation for the unknown and solve it.

b. Find the measure of ∠BDC.



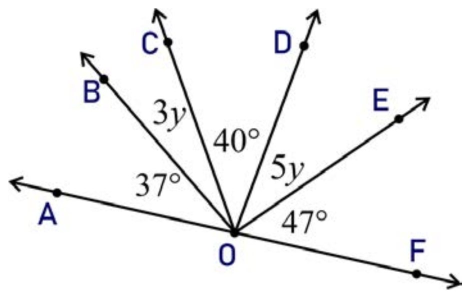
7. **a.** Write an equation for the unknown and solve it.

b. Find the measure of ∠ABC.



8. **a.** Write an equation for the unknown and solve it.

b. Find the measure of each angle in the image, excluding those whose angle measure is given.



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Problems Involving Circles

Sometimes, we might leave the answer in a format that includes π, and not give the answer as a rounded decimal. For example, if the diameter of a circle is 10 units, its circumference is 10π units, and its area is $\pi(5^2) = 25\pi$ square units. This format can be useful if there is a further calculation to be made using those values, and/or if we anticipate the π to cancel out in a further calculation.

Example. How much bigger in area is a circle with a radius 12 cm than a circle with a radius of 6 cm?

The former has an area of $\pi(12\text{ cm})^2 = 144\pi\text{ cm}^2$. The latter has an area of $\pi(6\text{ cm})^2 = 36\pi\text{ cm}^2$. To compare the two, we will write their ratio (using the fraction line):

$$\frac{\text{area of the smaller circle}}{\text{area of the bigger circle}} = \frac{36\pi\text{ cm}^2}{144\pi\text{ cm}^2} = \frac{36}{144} = \frac{1}{4}$$

Both the π and the unit "cm^2" cancel out, leaving the fraction 36/144 which then simplifies to 1/4. So, the area of the smaller circle is only 1/4 of the area of the larger.

If you do this calculation with decimals, you will have to round the intermediate answers, and thus the final answer will not be exactly 1/4 (though it will be close). Using π instead of rounded decimals will keep the calculations perfectly accurate.

You may use a calculator for all the problems in this lesson.

1. Tell from memory the formulas for the area of a circle and the circumference of a circle.
   If you have difficulty with this, work on memorising the formulas.

2. **a.** The circumference of a certain circle is 8π units. What is its diameter? Radius?

   b. The area of a certain circle is 36π square units. What is its radius? Its diameter?

3. The table gives the radii of three circles.

	Circle 1	**Circle 2**	**Circle 3**
Radius	5 cm	10 cm	15 cm

   a. What fraction is the area of Circle 1 of the area of Circle 2?

   b. What fraction is the area of Circle 1 of the area of Circle 3?

   c. What fraction is the area of Circle 2 of the area of Circle 3?

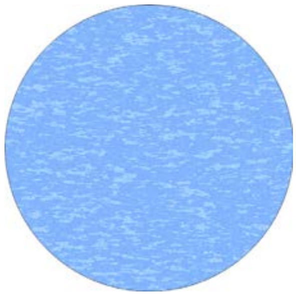
   d. (optional) How do the fractions above compare to the corresponding fractions formed from the radii of the circles?

4. The table shows the diameter and the circumference of some circular objects. One of the values is in error. Which?

Explain how you know.

Object	Diameter	Circumference
Bicycle tire	70 cm	220 cm
Pot lid	9 in	28 in
A pond	19 m	29 m
A ring	12 mm	38 mm
Circle A	5	5π

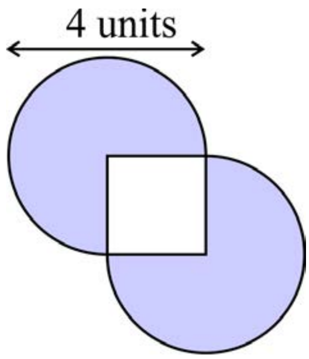
5. This is a scale drawing of a circular pond in a park, drawn here at the scale of 1:500.



a. Find the area of the pond in reality, to the nearest 10 square metres.

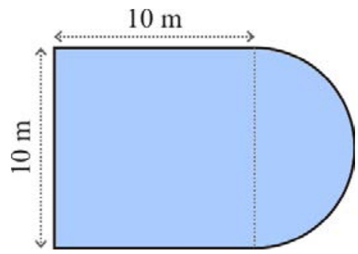
b. There will be a fence built around the pond. How long will the fence be, to the nearest metre?

6. Find the area of the shaded part of this circle design, in terms of π.



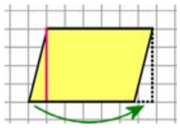
7. A swimming pool has the shape of the union of a square and a semicircle. The outer perimeter of the pool will be lined with decorative tiles that are 20 cm long. There will be 1 cm spaces between the tiles. The tiles are sold in packages of 20 that cost $24.90 per package.

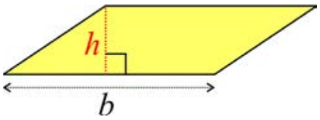
Find how many packages of tiles must be purchased to line the perimeter of the pool and the total cost.



Area of Polygons and Compound Shapes

Recall that from any parallelogram we can cut off a triangular piece and move it to the other side to make it a rectangle. This shows us that we can calculate the area of a parallelogram the same way as the area of a rectangle.

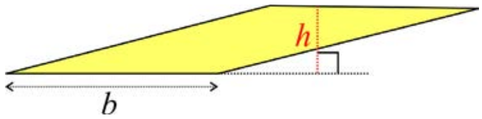




As a formula, the area of a parallelogram is $A = bh$

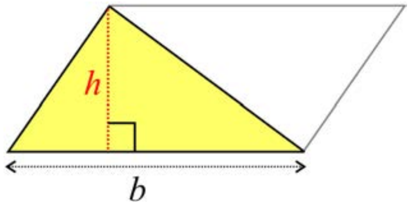
where b is the base and h is the altitude (height).

The **altitude** of a parallelogram is a perpendicular line segment from the base, or the extension of the base, to the top. Thus, the altitude might not be inside the parallelogram, if the parallelogram is very "slanted."

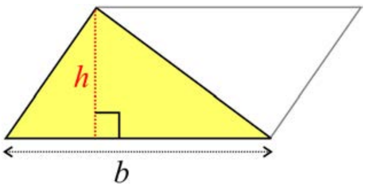


Since any triangle is half of its corresponding parallelogram, the area of a triangle is half the area of that parallelogram:

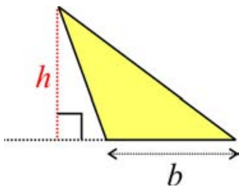
$A = \dfrac{bh}{2}$ where b is the base and h is the altitude of the triangle.



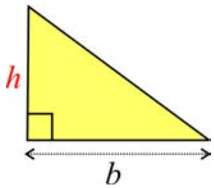
The **altitude** of a triangle is a line from one vertex to the opposite side that is perpendicular to that side. It can:



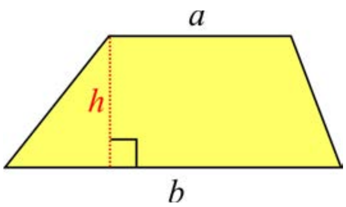
(1) fall inside the triangle;



(2) fall outside the triangle;



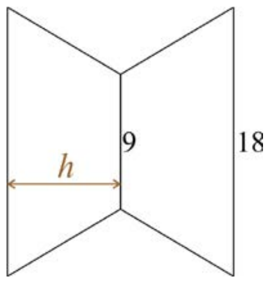
(3) be one of the sides of a right triangle.



The **area of a trapezium** is given by the formula $A = \dfrac{(a + b)}{2}h$

where a and b are the lengths of the two parallel sides and h is the altitude. Essentially, we calculate the average of the lengths of the two parallel sides, and multiply that times the height.

Example. What should be the height of these two identical trapeziums so that their combined area would be 189 square units?

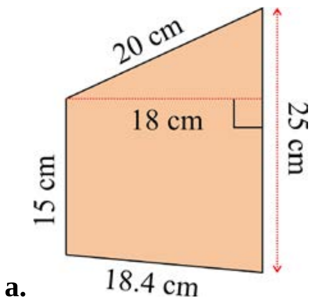


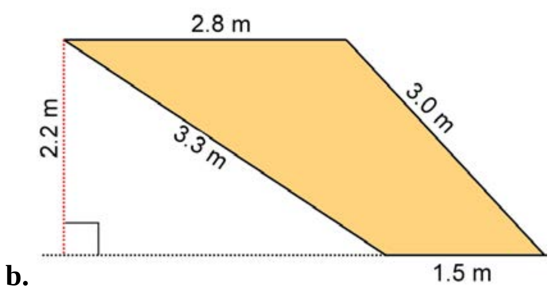
The area of each trapezium is $(9 + 18)/2 \cdot h$ which simplifies to $13.5h$.
The area of the two trapeziums is $2 \cdot 13.5h$ which simplifies to $27h$.

So, we simply solve the equation $27h = 189$, from which $h = 189/27 = 7$ units.

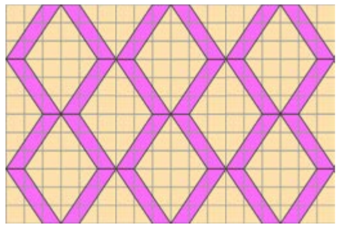
You may use a calculator for all the problems in this lesson.

1. Find the area of each trapezium.



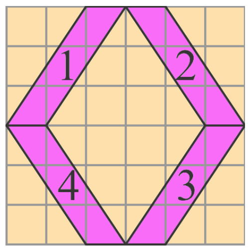


2. This is a pattern of light-coloured rhombi with dark pink borders.
The figure below shows the basic unit, or cell, of the pattern.
The pink borders actually consist of parallelograms.



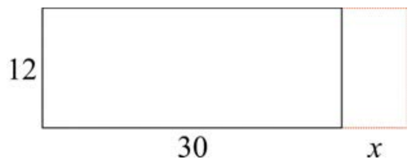
a. Find the area, in square units, of one of the numbered parallelograms.

b. Find the total area, in square units, of the lightly-coloured parts of the cell.



c. What percentage of the entire pattern do the rhombi (and parts of rhombi) cover?

3. A 12 × 30 rectangle is enlarged. How much longer should it be so that its area would be 444 square units?



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Probability

You *probably* already have an intuitive idea of what *probability* is. In this lesson we look at some simple examples in order to study probability from a mathematical point of view.

If we flip a coin, the chance, or **probability**, of getting "heads" is 1/2. The chance of getting "tails" is also 1/2. "Heads" and "tails" are the two possible **outcomes** when tossing a coin, and they are equally likely.

When rolling a six-sided number cube (a die), you have six possible **outcomes**: you can roll either 1, 2, 3, 4, 5, or 6. These are all equally likely (assuming the die is fair).

Thus the probability of rolling a five is 1/6. The probability of rolling a three is also 1/6. In fact, the probability of each of the six outcomes is 1/6.

The probability of rolling an even number is 3/6, or 1/2, because three of the six possible outcomes are even numbers.

Simple probability has to do with situations where each possible outcome is <u>equally likely</u>.

Then the **probability** of an event is the fraction $\dfrac{\textbf{number of favourable outcomes}}{\textbf{number of possible outcomes}}$

"favourable outcomes" are those that make up the event you want. The examples will make this clear.

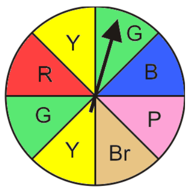
Example 1. What is the probability of getting a number that is less than 6 when tossing a fair die?

Count how many of the outcomes are "favourable" (less than 6). There are five: 1, 2, 3, 4, or 5.
And there are six possible outcomes in total.

Therefore, the probability is $\dfrac{\text{number of favourable outcomes}}{\text{number of possible outcomes}} = \dfrac{5}{6}$.

In maths notation we write "P" for probability and put the event in brackets: **P(less than 6) = 5/6.**

Example 2. On this spinner the number of possible outcomes is eight, because the arrow is equally likely to land on any of the eight wedges. What is the probability of spinning yellow?



There are TWO favourable outcomes (yellow areas) out of EIGHT possible outcomes.

P(yellow) = 2/8 = 1/4.

(Because green and yellow each have two wedges, there are only six possible colours that can result. When we list the possible outcomes, we list the six colours. However, when we figure the probabilities, we must use the eight equal-sized wedges to find the probability.)

By convention, the probability of an event is always at least 0 and at most 1. In symbols: $0 \le$ P(event) ≤ 1.

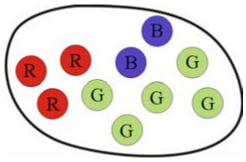
A probability of 0 means that the event does not occur; it is impossible. Probability of 1 means that the event is sure to occur; it is certain. A probability near 1 (such as 0.85) means that the event is likely to occur. A probability of 1/2 means that an event is neither likely nor unlikely.

Example 3. What is the probability of rolling 8 on a standard six-sided die?

This is an impossible event, so its probability is zero: P(8) = 0.

Example 4. What is the probability of rolling a whole number on a die?

This is a sure event, so its probability is one. P(whole number) = 1.



1. There are three red marbles, two dark blue marbles, and five light green marbles in Michelle's bag. List all the possible outcomes if you choose one marble randomly from her bag.

2. Michelle chooses one marble at random from her bag. What is the probability that...

   a. the marble is blue?

   b. the marble is not red?

   c. the marble is neither blue nor green?

3. Make up an event with a probability of zero in this situation.

4. Suppose you choose one letter randomly from the word "PROBABILITY."

   a. List all the possible outcomes for this event.

   Now find the probabilities of these events:

   b. P(B)

   c. P(A or I)

   d. P(vowel)

   e. Make up an event for this situation that is likely to occur, yet not a sure event, and calculate its probability.

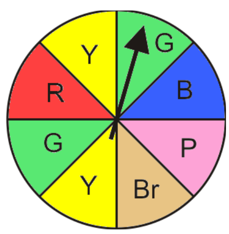
The complement of an event and the probability of "not"

The ***complement*** of any event A is the event that A does *not* occur.

If the probability of event A is a, then the probability of A not happening is simply $1 - a$.

5. The weatherman says that the chance of rain for tomorrow is 1/10. What is the probability of it not raining?

6. The spinner is spun once. Find the probabilities as simplified fractions.



   a. P(green) **b.** P(not green)

   c. P(not pink) **d.** P(not black)

   e. Make up an event for this situation that is not likely, yet not impossible either, and calculate its probability.

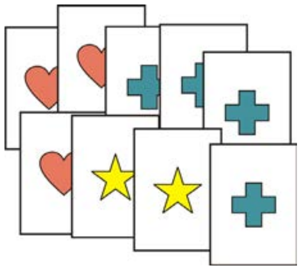
Probabilities are often given as percentages instead of fractions.

Example 5. Kimberly’s sock bin contains 7 brown socks, 9 white socks, and 5 red socks. She picks one without looking. What is the probability that she gets a white sock?

There are 9 white socks out of 21 socks in all. The probability is 9/21 = 3/7 ≈ 0.42857 = 0.429 = 42.9%.

7. Suppose you were to draw one card from the set of cards on the right. Complete the table with the possible outcomes, and their probabilities both as fractions and as percentages (to the nearest tenth of a percent).

Possible outcomes	Probability (fraction)	Probability (percentage)



8. This “rainbow spinner” is spun once. Find the probabilities to the nearest tenth of a percent.

a. P(yellow)

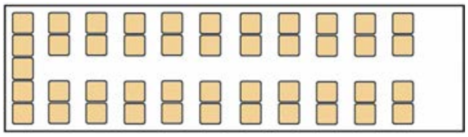
b. P(blue or green)

c. P(not orange)

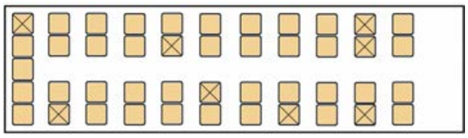
d. P(not red and not purple)

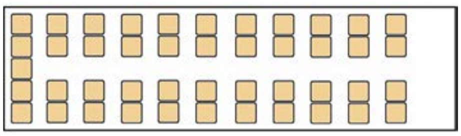
e. Make up an event for this situation with a probability of 1.

9. **a.** An empty bus has 45 seats, and 22 of them are window seats. If you are assigned a seat at random, what is the probability, to the nearest tenth of a percent, that you get a window seat?



b. Now each seat marked with an “x” is already occupied. If you choose a seat randomly, what is the probability, to the nearest tenth of a percent, that you get a window seat?





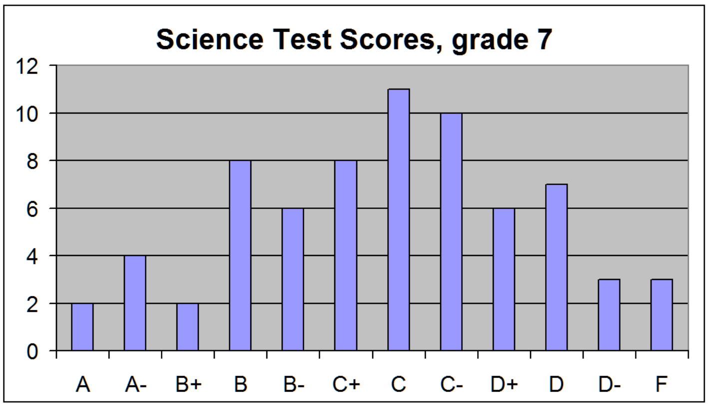
The chart shows you the seating arrangement of a bus. You enter the bus, and the driver informs you that fifteen seats are already occupied and that if you choose a seat randomly, the probability of getting a window seat is less than 25%.

How many window seats are occupied, at least?

Puzzle Corner

Probability Problems from Statistics

Example 1. The bar graph shows the science test scores of all seventy 7th graders in Westmont School. If you choose one of them at random, then what is the probability that the student's score was at least C− (in other words, C− or better)?



Sometimes when a probability question involves "at least," it is easier to look at the complement event — everything else — and find its probability first. The complement of "at least C–" is "at most D+" in other words, D+, D, D–, and F. From the graph, it is easier to sum the number of students who got the four low scores than to sum the number of students who got the eight high scores.

The number of students who got D+, D, D–, or F is 6 + 7 + 3 + 3 = 19 students. There are a total of 70 students, so P(at most D+) = 19/70. Now it's easy to calculate the original probability in question: P(at least C–) = 1 − 19/70 = 51/70.

1. You choose one student at random from the 7th graders in Westmont School.

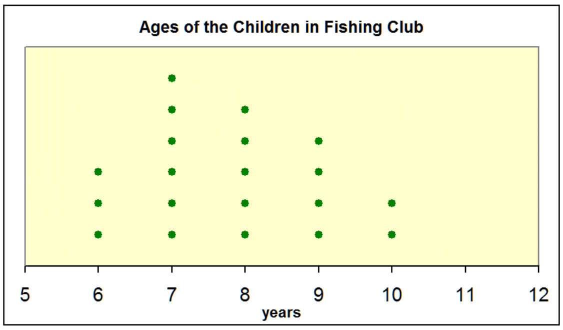
   a. What is the probability that the student's score was at least D?

   b. What is the probability that the student's score was at most B+?

2. The dotplot shows the age distribution of a children's fishing club. One child is chosen randomly from the group.

   a. What is the probability that the child is at most 9 years of age?

   b. What is the probability that the child is at least 7 years of age?



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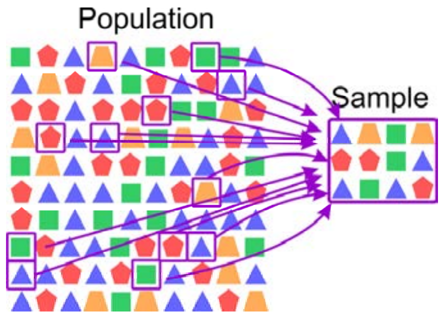
Random Sampling

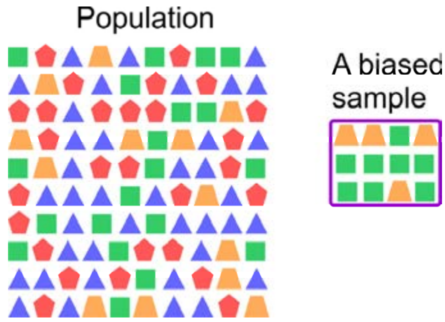
When researchers have a question concerning a large population, they obtain a **sample** (a part) of that population. That is because it is typically impossible to study the entire population.

For example, if you want to know how the citizens of France feel about climate change, you cannot just go and ask every person in France about it. You would choose for example 600 French citizens as your sample and ask them your question.



The way a sample is chosen is very important. Some methods of sampling may produce a sample that is *not* representative of the entire population. We call that a **biased sample**.

For example, if you are studying a student population of 630 in a school with close to an equal number of boys and girls, and you happen to choose a sample of 20 boys, then your sample is biased. It doesn't represent the entire population well.



We need to use **unbiased sampling methods** in order to get a sample that truly represents the population being studied. The best way to avoid biased samples is to select a **random sample.**

The main characteristics of a random sample are:

1. **Randomness:** each member of the population has an equal chance of being selected.

   Let's say a researcher is studying the types of cars Americans own. He decides to interview only people he finds at a local mall because that mall is close to where he lives, so it is convenient for him. His sample is biased because not every member of the US population even has a chance to be selected in his sample. Maybe the people at his local mall are predominantly rich people who own several cars per family, so in that respect those people would not be a good representation of the entire population of the US.

   We call this type of sample a **convenience sample** because it is convenient or easy to obtain.

2. **External selection:** respondents must be chosen by the researcher, not self-selected.

   If our researcher mails a questionnaire to various people across the US asking them to fill it out and return it, his sample is a **voluntary response sample**, which is a biased sample. Some people volunteer to return the questionnaire, but others don't. The people themselves decide whether or not to be a part of the sample.

   Why might this be a problem? Some of the people who would choose to take part may have an external reason to do so. They might want to show off how "good" they are in the particular aspect being studied, or they might just like to speak out about their opinions.

Our researcher could get a true random sample by choosing people randomly from a list of people living in the US and calling them. That way, each person has an equal chance of being selected in the sample (it is random), and the people cannot self-select to take part (the researcher chooses who takes part).

An unbiased sampling method is more likely to produce a representative sample.

1. You are studying whether students in a large college prefer to drink coffee black, with milk, with cream, or with sweetener, or whether they prefer not to drink coffee at all.

   a. Which of the six sampling methods listed below produce a voluntary response sample?

   b. Which methods don't give each member of the student population an equal chance to be selected for the sample?

   c. Which method is likely to produce a sample with only coffee drinkers, overlooking those who don't drink coffee?

   d. Which method will be the most likely to give you a representative (unbiased) sample?

   Sampling Methods

   (1) You interview 80 students in a cafe on the campus.

   (2) You interview 80 students who come in at the main door of the campus.

   (3) You interview the first 80 students you happen to meet on a certain day.

   (4) You choose 80 names randomly from a list of all the students. You call them to interview them.

   (5) You send an email to all the students in the college, asking them to fill in a form on a web page you have set up. You hope to get at least 80 responses.

   (6) You choose 80 names randomly from a list of all the students. You send them an email, asking them to fill in a form on a web page you have set up.

2. A recipe website posts a poll on their home page that any visitor to that website can take. In it, they ask if people are looking for a recipe for a dessert, a main dish, a side dish, bread, or salad. During the course of one Sunday, 4600 people visit the page, and 252 of them fill in the poll. Explain why the poll results will be based on a biased sample.

Some common random sampling methods are:

1. **Simple random sampling.** Each individual in the sample is chosen randomly and entirely by chance, perhaps by using dice, through pulling names out of a hat, or with a random number generator.

2. **Systematic random sampling.** The individuals of the population are placed in some order, and then each individual at a certain specified interval is selected for the sample.

   For example, a supermarket might study the shopping habits of its customers by choosing every 15th customer who enters the store for the sample.

3. **Stratified random sampling.** The population is first divided into categories (strata) and then a random sample is obtained from each category.

   For example, to study how much sleep students in a particular school get, you might first divide the students into groups by grade levels (the stratification), then select a random sample from each of the grade levels.

3. A population to be studied doesn't have to be of people. A factory produces MP3 players. Out of the 500 units that the factory produces each day, a quality control inspector selects 25 for testing to study their quality and reliability. Which way should he choose those 25 so that his sample would best represent all the MP3 players that the factory produces?

   a. Choose the first 25 produced on a given day.

   b. First choose a number between 1 and 20 randomly. Select the player corresponding to that number, and after that, every 20th player, in the order they were produced that day.

   c. Choose 25 players that have just been finished around 1 PM when the inspector is touring the factory.

   d. Generate 25 random numbers between 1 and 500 and choose the corresponding 25 MP3 players in the order they were produced that day.

4. Ryan has two large fields planted with green beans. He wants to compare the bean plants in one field with the plants in the other. Design a practical sampling method for him to produce an unbiased sample.

Sometimes it is not obvious how a particular sampling method might be biased.

If you are studying students' homework habits in a particular school, it might initially make sense to interview the first 25 students who come into the school in the morning. However, there could be an underlying factor that makes that method biased. What if students who are diligent with their homework also tend to come to school early? In that case, students who are not diligent don't have an equal chance of being selected for your sample. A better method is to use systematic random sampling and to choose, say, every 10th student entering the school for the sample.

5. Heather is studying the effect of how the method of feeding affects the health of a baby during its first year of life. She has already determined that babies who are fed with infant formula get sick more often than babies who are fed with human milk, but she especially wants to find out how often babies who are fed both formula and breast milk get sick. Explain why interviewing mothers in the places below will produce a biased sample:

   a. A pediatrician's office.

   b. A breastfeeding class for new mums.

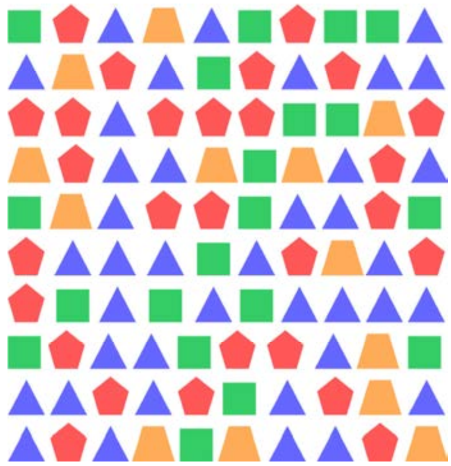
6. **a.** Devise a method that will produce a biased sample based on self-selection, and explain how that would happen, based on Heather's situation in Question 5.

   b. Design a sampling method for Heather that is most likely to produce a representative sample.

7. An organisation that helps teenagers with drug problems has set up a telephone hot line for teens to call in to discuss their problems. After a few months of operations, the organisation wants to evaluate the effectiveness of their service. Since they don't usually get as many calls on Tuesdays, they decide to choose a particular Tuesday to ask each teen at the end of the call to answer a few questions about how the service has helped. Is this a good method for selecting a sample? Explain.

Using Random Sampling

1. In this activity, you will make several samples of 10 from this population of shapes:



   Since the shapes are in a 10 by 10 grid arrangement, you can easily assign a number from 1 to 100 for each shape. To obtain a random sample, you can use one of these ideas or come up with your own.

   - Choose a random number between 1 and 10. Then, starting from that number, choose every 10th shape.
   - Go to **https://www.random.org/integers/** and generate 10 random integers between 1 and 100. (If the set of numbers contains a duplicate, discard that set and make another.)

Here is an example sample (Sample 1) to help you get started:

It is based on generating these random numbers at the website above: `76 17 51 63 88 29 95 73 40 69`

Generate at least five more samples. Count the number of each kind of shape in each of your samples, and fill in the table. Lastly, calculate the average number of triangles, the average number of squares, and so on.

	Sample 1	Sample 2	Sample 3	Sample 4	Sample 5	Sample 6	Averages
triangles	5						
squares	1						
pentagons	3						
trapeziums	1						
Total	10	10	10	10	10	10	—

2. Now imagine that you haven't seen the entire population of shapes but you try to infer (conclude) something about the entire population of shapes based on these six samples.

   a. Which shape seems to be the most common?

   b. Which shape seems to be the least common?

   c. List the shapes in order, from the least common to the most common:

   d. Based on the average number of the shapes in the six samples, *estimate* how many of each shape there are in the entire population of 100 shapes:

   _____ triangles   _____ squares   _____ pentagons   _____ trapeziums

Here are some important points to realise and remember concerning random sampling. You probably noticed these facts while doing the activity:

1. **Random samples vary.** The differences occur simply because their members are indeed chosen from the population at random.
2. **A random sample is more likely to be unbiased, but that is not 100% certain.** In other words, it is possible for a random sample *not* to be representative of the population as a whole. However, the chances of a random sample being unbiased are greater than the chances of a non-random sample being unbiased.
3. **It is better to base inferences about the population on more than one sample.** However, if you cannot obtain more than one sample, then it is recommended to increase the sample size as much as possible, because a large sample represents the whole population better than a small one.

Example 1. Three people are running for mayor in a town with 20 000 voters. Two companies conducted separate polls of 350 people, asking who they would vote for in the final election. Here are the results:

	Smith	Harrison	Jones
Poll 1	63	220	67
Poll 2	53	238	59

Based on these results, what can we conclude about the results of the final election?

In both polls, Harrison is winning and by a large margin. In other words, far more people are claiming that they will vote for Harrison than for Smith or Jones. So we can fairly confidently conclude that <u>Harrison will be the winner of the actual election</u>.

Not only that, but in both polls Jones did better than Smith. So <u>it is likely, but not sure, that Jones will beat Smith in the actual election</u>. We cannot say that for sure because the differences are small: 4 and 6 votes.

We can also quantify the election results (use actual numbers). In Poll 1, 220 is more than three times 63 or 67. In Poll 2, 238 is more than four times 53 and about four times 59. Based on those, we can say that <u>Harrison will get roughly 3-4 times more votes than either of his opponents</u>.

Example 2. The data below presents the results of three different samples from a study about how students prefer to drink coffee.

	Black	With milk	With cream	Milk and sugar	Cream and sugar	Totals
Sample 1	12	21	24	36	37	130
Sample 2	9	23	22	37	39	130
Sample 3	14	18	20	36	42	130

What can we infer based on the data?

(1) Looking at the numbers carefully we can see that in each of the samples “Cream and sugar” was the winner and “Milk and sugar” came fairly close behind it.

(2) The two options “With milk” and “With cream” are also close to each other, but we cannot say for sure which of them is preferred, because in Samples 1 and 3, “With cream” beats “With milk”, whereas in sample 2 it is the opposite way.

(3) Drinking black coffee is the least popular option in all three samples.

(4) We can quantify the results. For example, “Cream and sugar” is the favourite of roughly 40/130 = 4/13 of the students, and 4/13 is almost 4/12 = 1/3. So we can state that <u>almost 1/3 of the students prefer to drink their coffee with cream and sugar</u>. You can make similar statements using approximate fractions for the other options.

What kinds of inferences can you make about the entire population based on random samples?

Based on what the data demonstrates, you may be able to…

- state which option is the most or least, the best or worst, the winner or loser, *etc*.
- compare two options as better or worse, more or less, *etc*.
- quantify the above statements with numbers, fractions, or percentages:
  How much more or less is one option than another?
- find trends: identify an increase or a decrease in some quantity as some other quantity, such as time, increases or decreases.

3. A large workplace conducted a survey of their employees' sleeping hours. They took two samples of 65 people, one week apart. What can you infer based on these results?

	< 5 h	5 h	6 h	7 h	8 h	> 8h	Totals
Sample 1	1	4	21	32	6	1	65
Sample 2	2	8	23	26	4	2	65

4. A music band wanted to find out which of their songs their audience likes best. They randomly chose some people to be interviewed after two of their concerts, asking them what their favourite song was. The results are in the table at the right.

   What conclusions can you draw from the data?

Songs	**Concert 1 (Sample 1)**	**Concert 2 (Sample 2)**
"Love You"	4	3
"My Best"	9	11
"Never Again"	7	5
"Sunshine"	5	6
Totals	25	25

Example 3. Let's go back to our example about the three candidates running for mayor. If 15 000 people end up voting for mayor in the actual election, estimate the number of votes each candidate will get.

First we calculate the percentage of votes each candidate got in each poll:

	Smith	Harrison	Jones	Total
Poll 1	63	220	67	350
Percentage	18.00%	62.86%	19.14%	100%
Poll 2	53	238	59	350
Percentage	15.14%	68.00%	16.86%	100%
Averages:	**16.57%**	**65.43%**	**18.00%**	**100%**

Smith: Based on Poll 1, we would estimate that he would get 0.18 · 15 000 = 2700 votes. Based on Poll 2, our estimate would be 0.1514 · 15 000 = 2271 votes. Let's use the average percentage of 16.57% based on both polls. We estimate he will get around 0.1657 · 15 000 = 2486 votes ≈ <u>2500 votes</u>.

Then we can use the estimates from the individual polls to gauge how far off our estimate of 2500 votes is. These numbers (2700 and 2271 votes) differ from 2500 votes by about 200-300 votes. Since the samples are random, the estimate of 2500 votes may be off by 2-3 times the 200-300 votes; we cannot really know without having dozens of samples. So, we state that our estimate may be <u>off by several hundred votes</u>.

Harrison: Based on Poll 1, he would get 0.6286 · 15 000 = 9429 votes. Based on Poll 2, he would get 0.68 · 15 000 = 10 200 votes. Using the average percentage, we estimate he will get around 0.6543 · 15 000 = 9815 votes ≈ <u>9800 votes</u>.

The estimates of 9429 and 10 200 votes differ from the estimate of 9800 votes by about 400 votes. Again, the real value may differ from the estimate several times the 400 votes. Based on that, we gauge that the estimate of 9800 votes may be <u>off by over a thousand votes</u>.

Jones: Based on Poll 1, he would get 0.1914 · 15 000 = 2871 votes. Based on Poll 2, he would get 0.1686 · 15 000 = 2529 votes. Using the average percentage, we estimate he will get around 0.18 · 15 000 = <u>2700 votes</u>.

The two estimates of 2871 votes and 2529 votes differ from that by a few hundred votes. We might gauge that the estimate of 2700 votes may be <u>off by hundreds of votes</u>.

5. **a.** Let's continue quantifying the results of the study about how students prefer to drink coffee. Use statements with fractions, such as "about 1/5" and "slightly more/less than 1/10."

	Black	With milk	With cream	Milk and sugar	Cream and sugar	Totals
Sample 1	12	21	24	36	37	130
Sample 2	9	23	22	37	39	130
Sample 3	14	18	20	36	42	130

_________________________________ of the students prefer to drink coffee black.

_________________________________ of the students prefer to drink coffee with milk but no sugar.

_________________________________ of the students prefer to drink coffee with cream but no sugar.

_________________________________ of the students prefer to drink coffee with milk and sugar.

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Some Revision

In this lesson we revise how to make boxplots and how to calculate mean absolute deviation of a data set.

A **boxplot** (also called a box-and-whisker plot) is a handy way to show graphically how the data is spread and where its centre (median) is. It can be drawn horizontally or vertically. To draw a boxplot, we use the **five-number summary**, which consists of the minimum of the data plus the four quartiles. The four quartiles divide the data into quarters:

- The first (or lower) quartile is the median of the lower half of the data.
- The second quartile is the median.
- The third (or upper) quartile is the median of the upper half of the data.
- The fourth quartile is the maximum of the data.

Example 1. Let's make a dot plot and a boxplot of the ages of the children in a softball club. Their ages are: 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 7, 7, 7, 8, 8, 8, 8, 8, 9, 9, 11.

For the boxplot, we need to determine the median and the 1st and 3rd quartiles. This means we divide the data into quarters... but first, we divide it into upper and lower halves. Since there is an *odd* number of children, we will leave the median age (7) by itself, so it belongs to *neither* the lower nor the upper half.

4, 4, 5, 5, 5, 6, 6, 6, 7, 7, **7**, 7, 7, 8, 8, 8, 8, 8, 9, 9, 11

The first quartile is the median of the lower half of the data, which is the average of 5 and 6, or <u>5.5 years old</u>. Similarly, the third quartile is the median of the upper half of the data, <u>8 years old</u>.

4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 7, 7, 7, 8, 8, 8, 8, 8, 9, 9, 11

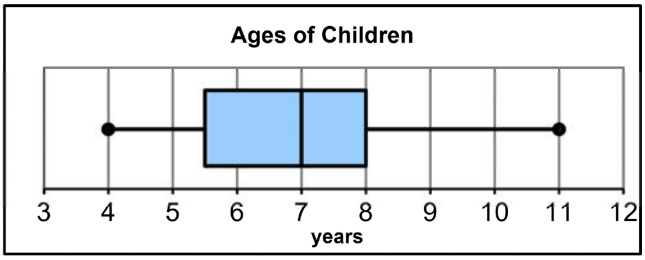
1st quartile: 5.5 median 3rd quartile: 8

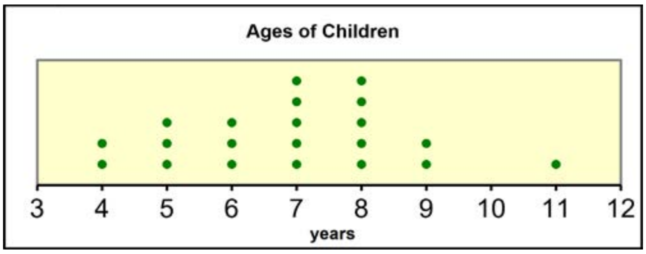
The **interquartile range (IQR)** is the difference between those two: 8 years − 5.5 years = <u>2.5 years</u>. It is a measure of spread. It shows us that 50% of the data is within an interval of 2.5 years (from 5.5 to 8 years). In other words, half of the children are from 5.5 to 8 years old. The bigger the IQR, the more the data is spread out.

Note that we include the unit (in this case "years old" or "years") with all of these quantities.

Now we can draw the boxplot. Below it, the same data is shown in a dot plot. Compare the two!

- The first "whisker" represents the first quarter of the data, or these ages: 4, 4, 5, 5, 5. The box represents the middle half of the data and includes the median: (6, 6, 6, 7, 7, **7**, 7, 7, 8, 8, 8)
- The last whisker represents the last quarter of the data: 8, 8, 9, 9, 11.
- Since the second half of the box is fairly short, the data is concentrated there: there are a lot of 7 and 8-year olds.





1. Read the five-number summary from the boxplot, and give the interquartile range.

Minimum:

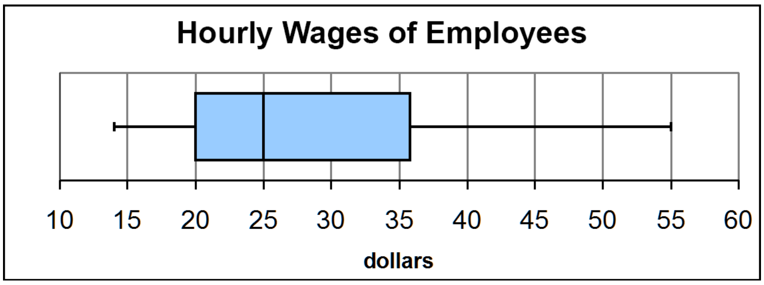
First quartile:

Median:

Third quartile:

Maximum:

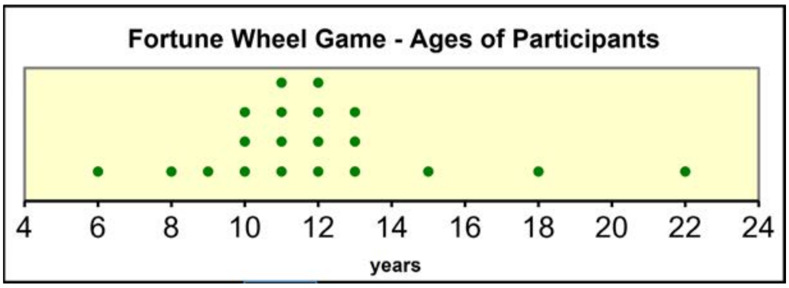
Interquartile range:

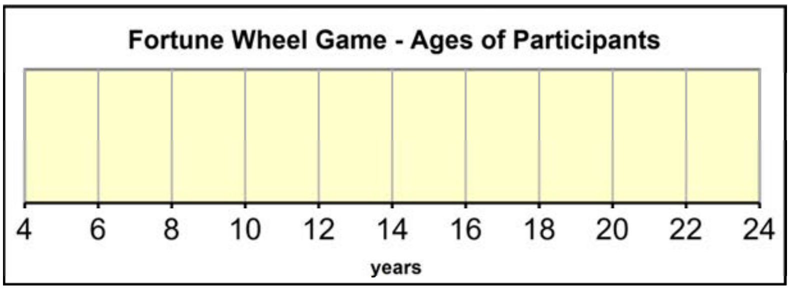


2. Below you see the ages of participants in a fortune wheel game, and a dotplot made from the data:

6, 8, 9, 10, 10, 10, 11, 11, 11, 11, 12, 12, 12, 12, 13, 13, 13, 15, 18, 22

a. Calculate the five-number summary from the data, and draw a boxplot below the dot plot.





Five-number summary

Minimum ________

First quartile ________

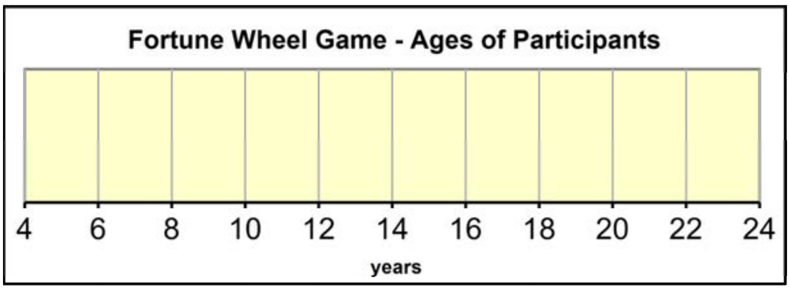
Median ________

Third quartile ________

Maximum ________

Interquartile range ________

b. Let's say on another day, a different set of people came to play the fortune wheel game. Draw another boxplot to represent that group of people, so that its median is 14.5 years, the IQR is 2.5, and the range is 11 years. (Note that there are many possible ways to do this.)



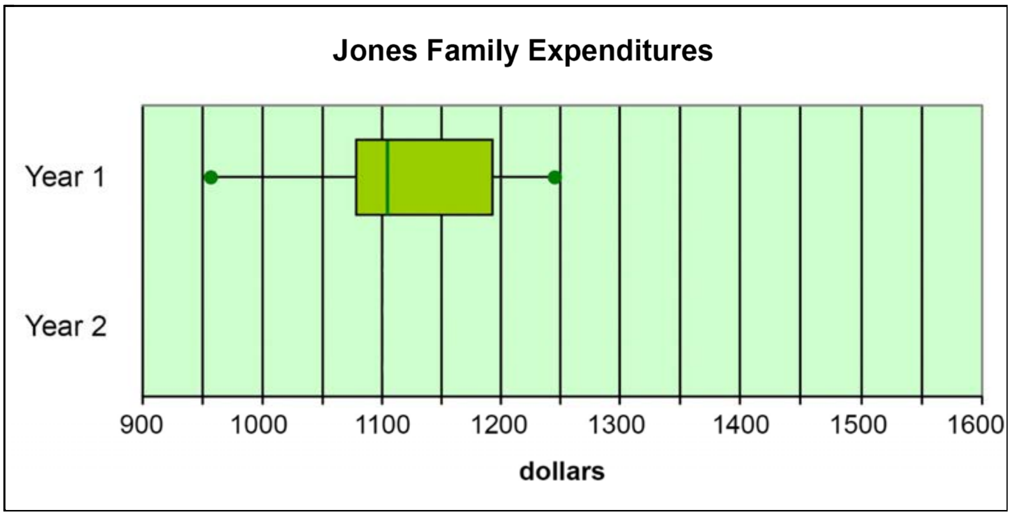
c. Which group had older participants? How do you know?

d. Which group is more varied in its ages? How do you know?

3. The data below shows the Jones family's monthly expenditures for 12 months. The data is already organised in ascending order and divided into quarters; the first number, $956, is not necessarily for January.

$956 $999 $1076 $1080 $1086 $1100 $1110 $1165 $1190 $1196 $1245 $1245

Here is a boxplot made from the data:



Let's say that the following year (Year 2), the Jones family's expenditures were these (again, already in ascending order):

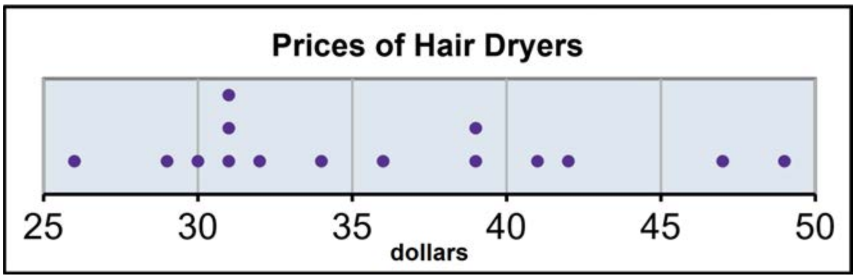
$925 $949 $1090 $1250 $1288 $1360 $1420 $1465 $1480 $1520 $1575 $1590

a. Draw a boxplot for this data, below the Year 1 boxplot in the image above.

b. In which year did their monthly expenditures vary more?

In which year were their monthly expenditures more, overall?

4. Make a boxplot using the data in the dot plot.



Mean absolute deviation (MAD for short) is a measure of variability. It tells us how much the individual data items differ from their mean, on average. The term “mean absolute deviation” actually explains itself:

- *Deviation* is a difference: we look at how much the individual data values deviate (differ) from the mean.
- *Absolute* deviation means the absolute value of each deviation. We take all of the deviations (differences) as positive.
- *Mean* absolute deviation is the mean (average) of all the absolute deviations.

Example 2. Calculate the mean absolute deviation for this data set: 5, 6, 6, 6, 6, 7, 7, 8, 8, 9.

First we calculate the mean itself:
Mean = (5 + 6 + 6 + 6 + 6 + 7 + 7 + 8 + 8 + 9)/10 = 6.8.

Next, we calculate how much each data value deviates from the mean of 6.8. The table on the right lists the absolute deviations.

Lastly, we calculate the mean of the absolute deviations:
MAD = (1.8 + 4 · 0.8 + 2 · 0.2 + 2 · 1.2 + 2.2)/10 = 1.0

So the mean absolute deviation is 1.0. This means that, on average, the individual data values differ from the mean of 6.8 by 1. This value (1.0) is not especially large compared to the mean (6.8), so the data is fairly concentrated (not spread out). We can even see that from the list of data values.

Value	Absolute deviation from the mean
5	1.8
6	0.8
6	0.8
6	0.8
6	0.8
7	0.2
7	0.2
8	1.2
8	1.2
9	2.2
mean 6.8	**MAD 1.0**

5. **a.** Calculate the mean and MAD for this data, to one decimal digit:
85 90 91 96 105 114 117 120 138 209 299.

However, when calculating the MAD, don’t use the mean rounded to one decimal digit. You need some extra decimals for the intermediate calculations, so use a mean with three decimal digits.

b. Let’s say these are prices of some gadget, in different stores. Would you say that the prices vary a lot, a little, or neither?

c. Recall that the MAD tells us by how much the individual values vary from the mean, on average.

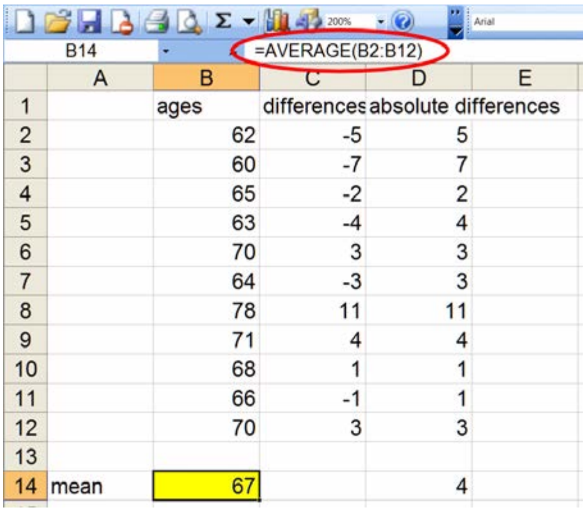
So, here, the prices vary from the mean of $__________

by $________, on average.

Would you say that reflects a small or large variability in the prices?

Value	Absolute deviation from the mean
85	
90	
91	
96	
105	
114	
117	
120	
138	
209	
299	
mean ________	**MAD ________**

It is much quicker to calculate the MAD using a spreadsheet program. For your reference, here are the instructions for how to do it in Excel. The process should be the same for LibreOffice Calc.

1. 

B14 =AVERAGE(B2:B12)

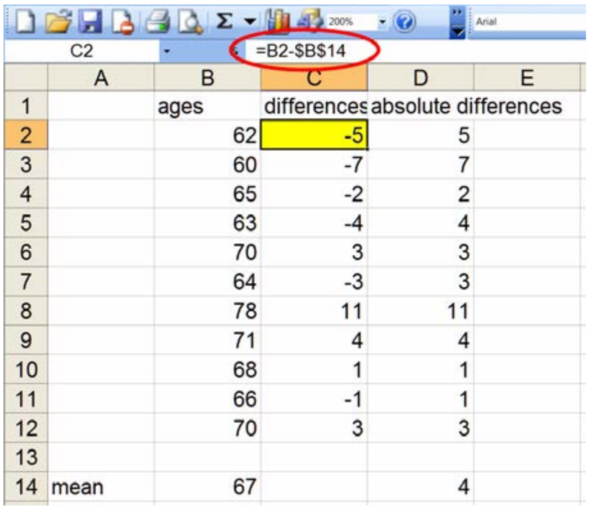
	A	B	C	D	E
1		ages	differences	absolute differences	
2		62	-5	5	
3		60	-7	7	
4		65	-2	2	
5		63	-4	4	
6		70	3	3	
7		64	-3	3	
8		78	11	11	
9		71	4	4	
10		68	1	1	
11		66	-1	1	
12		70	3	3	
13					
14	mean	67		4	

To calculate the mean of a set of data, in the cell where you want the calculation to appear, type:

=AVERAGE(B2:B12)

When you type the formula in the cell, it appears in the formula bar at the top, as in the image. A formula always starts with an equals sign.

Press "ENTER" to see the answer, 67.

2. 

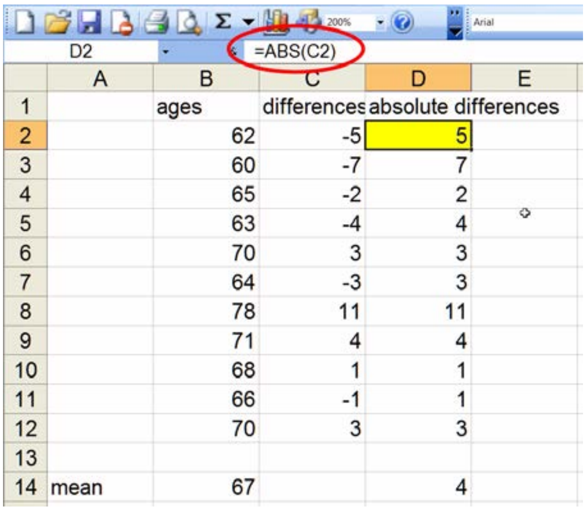
C2 =B2-B14

	A	B	C	D	E
1		ages	differences	absolute differences	
2		62	-5	5	
3		60	-7	7	
4		65	-2	2	
5		63	-4	4	
6		70	3	3	
7		64	-3	3	
8		78	11	11	
9		71	4	4	
10		68	1	1	
11		66	-1	1	
12		70	3	3	
13					
14	mean	67		4	

Next we calculate the difference between each item of data and the mean.

Type "=B2 - B14" to subtract the values in cells B2 and B14.

The dollar signs in B14 make it an **absolute reference**, so it doesn't change when you copy and paste the formula into other cells. Pasting the formula into the cells below is a quick way to get the spreadsheet to calculate those values, too.

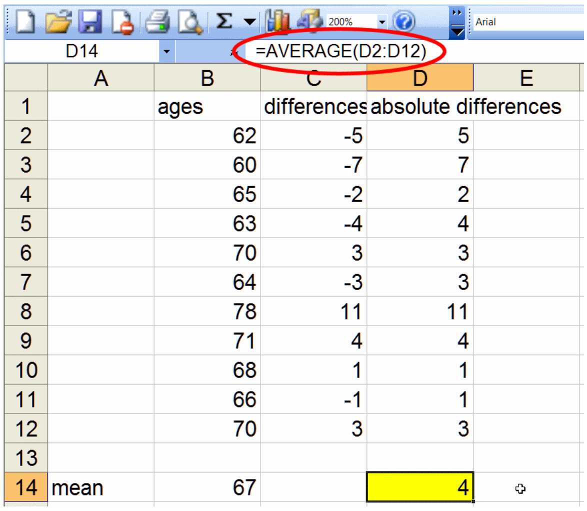
3. 

D2 =ABS(C2)

	A	B	C	D	E
1		ages	differences	absolute differences	
2		62	-5	5	
3		60	-7	7	
4		65	-2	2	
5		63	-4	4	
6		70	3	3	
7		64	-3	3	
8		78	11	11	
9		71	4	4	
10		68	1	1	
11		66	-1	1	
12		70	3	3	
13					
14	mean	67		4	

Now we calculate the absolute value of each difference.

In cell D2 type "=ABS(C2)" to calculate the absolute value of the number in cell C2. When you paste cell D2 into the cells below it, Excel automatically changes the reference to cell C2 to the correct cell.

4. 

D14 =AVERAGE(D2:D12)

	A	B	C	D	E
1		ages	differences	absolute differences	
2		62	-5	5	
3		60	-7	7	
4		65	-2	2	
5		63	-4	4	
6		70	3	3	
7		64	-3	3	
8		78	11	11	
9		71	4	4	
10		68	1	1	
11		66	-1	1	
12		70	3	3	
13					
14	mean	67		4	

Lastly, we are ready to calculate the mean absolute deviation by taking the average of the values in cells D2 to D12. In the cell where you want the value to appear, type "=AVERAGE(D2:D12)".

The answer "4" will then appear in the cell after you press "ENTER."

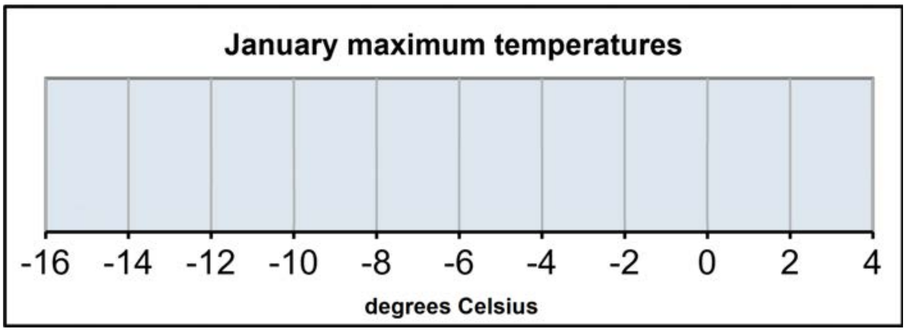
6. The table lists the highest daily temperatures for January.

Mon	Tue	Wed	Thu	Fri	Sat	Sun
−2°C	−4°C	2°C	3°C	1°C	−6°C	−5°C
0°C	−3°C	−3°C	−2°C	−10°C	−12°C	−15°C
−1°C	1°C	3°C	2°C	0°C	−2°C	−8°C
−12°C	−11°C	−9°C	−10°C	−5°C	−3°C	−4°C
−2°C	−2°C	0°C				

a. Calculate the mean to one decimal digit.

b. Calculate the mean absolute deviation (MAD) to one decimal digit. Use spreadsheet software if available.

c. Make a dot plot.



d. Fill in: The data values differ from the mean of _______ by ________, on average.

e. What is the median? Would median depict the centre of this data distribution better than the mean?