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Foreword

Math Mammoth Grade 7 comprises a complete math curriculum for the seventh grade mathematics studies. It follows the Common Core Mathematics Standards (CCS) for 7th grade. Those standards are so constructed that students can continue to a traditional algebra 1 curriculum after studying this. However, you also have the option of following this course with Math Mammoth Grade 8, which provides a gentler and slower transition to high school math.

The main areas of study in Math Mammoth Grade 7 are:

- The basics of algebra (expressions, linear equations and inequalities);
- The four operations with integers and with rational numbers (including negative fractions and decimals);
- Ratios, proportions, and percent;
- Geometry;
- Probability and statistics.

This book, 7-A, covers the language of algebra (chapter 1), integers (chapter 2), one-step equations (chapter 3), rational numbers (chapter 4), and equations and inequalities (chapter 5). The rest of the topics are covered in the 7-B worktext.

I heartily recommend that you read the full user guide in the following pages.

I wish you success in teaching math!

Maria Miller, the author

User Guide

Note: You can also find the information that follows online, at **https://www.mathmammoth.com/userguides/** .

Basic principles in using Math Mammoth Complete Curriculum

Math Mammoth is mastery-based, which means it concentrates on a few major topics at a time, in order to study them in depth. The two books (parts A and B) are like a “framework”, but you still have a lot of liberty in planning your student’s studies. You can even use it in a *spiral* manner, if you prefer. Simply have your student study in 2-3 chapters simultaneously.

In seventh grade, the first five chapters should be studied in order. Also, chapter 7 (Percent) should be studied before studying any of the chapters 8, 9, or 10 (Geometry, Probability, Statistics). You can be flexible with the rest of the scheduling.

Math Mammoth is not a scripted curriculum. In other words, it is not spelling out in exact detail what the teacher is to do or say. Instead, Math Mammoth gives you, the teacher, various tools for teaching:

- **The two student worktexts** (parts A and B) contain all the lesson material and exercises. They include the explanations of the concepts (the teaching part) in blue boxes. The worktexts also contain some advice for the teacher in the “Introduction” of each chapter.

  The teacher can read the teaching part of each lesson before the lesson, or read and study it together with the student in the lesson, or let the student read and study on his own. If you are a classroom teacher, you can copy the examples from the “blue teaching boxes” to the board and go through them on the board.

- There are hundreds of **videos** matched to the curriculum available at https://www.mathmammoth.com/videos/ . There isn’t a video for every lesson, but there are dozens of videos for each grade level. You can simply have the author teach your student!

- Don’t automatically assign all the exercises. Use your judgment, trying to assign just enough for your student’s needs. You can use the skipped exercises later for review. For most students, I recommend to start out by assigning about half of the available exercises. Adjust as necessary.

- For each chapter, there is a **link list to various free online games** and activities. These games can be used to supplement the math lessons, for learning math facts, or just for some fun. Each chapter introduction (in the student worktext) contains a link to the list corresponding to that chapter.

- The student books contain some **mixed review lessons**, and the curriculum also provides you with additional **cumulative review lessons**.

- There is a **chapter test** for each chapter of the curriculum, and a comprehensive end-of-year test.

- The **worksheet maker** allows you to make additional worksheets for most calculation-type topics in the curriculum. This is a single html file. You will need Internet access to be able to use it.

- You can use the free online exercises at https://www.mathmammoth.com/practice/
  This is an expanding section of the site, so check often to see what new topics we are adding to it!

- Some grade levels have **cut-outs** to make fraction manipulatives or geometric solids.

- And of course there are answer keys to everything.

How to get started

Have ready the first lesson from the student worktext. Go over the first teaching part (within the blue boxes) together with your student. Go through a few of the first exercises together, and then assign some problems for your student to do on their own.

Repeat this if the lesson has other blue teaching boxes. Naturally, you can also use the videos at https://www.mathmammoth.com/videos/

Many students can eventually study the lessons completely on their own — the curriculum becomes self-teaching. However, students definitely vary in how much they need someone to be there to actually teach them.

Pacing the curriculum

Each chapter introduction contains a suggested pacing guide for that chapter. You will see a summary on the right. (This summary does not include time for optional tests.)

Worktext 7-A	
Chapter 1	8 days
Chapter 2	13 days
Chapter 3	9 days
Chapter 4	16 days
Chapter 5	16 days
TOTAL	**62 days**

Worktext 7-B	
Chapter 6	17 days
Chapter 7	12 days
Chapter 8	23 days
Chapter 9	10 days
Chapter 10	12 days
TOTAL	**74 days**

Most lessons are 3 or 5 pages long, intended for one day. Some 5-page lessons can take two days. There are also a few optional lessons.

It can also be helpful to calculate a general guideline as to how many pages per week the student should cover in order to go through the curriculum in one school year.

The table below lists how many pages there are for the student to finish in this particular grade level, and gives you a guideline for how many pages per day to finish, assuming a 180-day (36-week) school year. The page count in the table below *includes* the optional lessons.

Example:

Grade level	School days	Days for tests and reviews	Lesson pages	Days for the student book	Pages to study per day	Pages to study per week
7-A	81	9	197	72	2.74	13.7
7-B	99	10	244	89	2.74	13.7
Grade 7 total	180	19	441	161	2.73	13.7

The table below is for you to use.

Grade level	School days	Days for tests and reviews	Lesson pages	Days for the student book	Pages to study per day	Pages to study per week
7-A			197			
7-B			244			
Grade 7 total			441			

Let’s say you determine that your student needs to study about 2.5 pages a day, or 12-13 pages a week on average in order to finish the curriculum in a year. As the student studies each lesson, keep in mind that sometimes most of the page might be reserved for workspace to solve equations or other problems. You might be able to cover more than the average number of pages on such a day. Some other day you might assign the student only one page of word problems.

Now, something important. Whenever the curriculum has lots of similar practice problems (a large set of problems), feel free to **only assign 1/2 or 2/3 of those problems**. If your student gets it with less amount of exercises, then that is perfect! If not, you can always assign the rest of the problems for some other day. In fact, you could even use these unassigned problems the next week or next month for some additional review.

In general, seventh graders might spend 45-90 minutes a day on math. If your student finds math enjoyable, they can of course spend more time with it! However, it is not good to drag out the lessons on a regular basis, because that can then affect the student's attitude towards math.

Working space, the usage of additional paper and mental math

The curriculum generally includes working space directly on the page for students to work out the problems. However, feel free to let your students to use extra paper when necessary. They can use it, not only for the "long" algorithms (where you line up numbers to add, subtract, multiply, and divide), but also to draw diagrams and pictures to help organize their thoughts. Some students won't need the additional space (and may resist the thought of extra paper), while some will benefit from it. Use your discretion.

Some exercises don't have any working space, but just an empty line for the answer (e.g. 200 + ______ = 1,000). Typically, I have intended that such exercises to be done using MENTAL MATH.

However, there are some students who struggle with mental math (often this is because of not having studied and used it in the past). As always, the teacher has the final say (not me!) as to how to approach the exercises and how to use the curriculum. We do want to prevent extreme frustration (to the point of tears). The goal is always to provide SOME challenge, but not too much, and to let students experience success enough so that they can continue enjoying learning math.

Students struggling with mental math will probably benefit from studying the basic principles of mental calculations from the earlier levels of Math Mammoth curriculum. To do so, look for lessons that list mental math strategies. They are taught in the chapters about addition, subtraction, place value, multiplication, and division. My article at https://www.mathmammoth.com/lessons/practical_tips_mental_math also gives you a summary of some of those principles.

Using tests

For each chapter, there is a **chapter test**, which can be administered right after studying the chapter. **The tests are optional.** Some families might prefer not to give tests at all. The main reason for the tests is for diagnostic purposes, and for record keeping. These tests are not aligned or matched to any standards.

The end-of-year test is best administered as a diagnostic or assessment test, which will tell you how well the student remembers and has mastered the mathematics content of the entire grade level.

Using cumulative reviews and the worksheet maker

The student books contain mixed review lessons which review concepts from earlier chapters. The curriculum also comes with additional cumulative review lessons, which are just like the mixed review lessons in the student books, with a mix of problems covering various topics. These are found in their own folder in the digital version, and in the Tests & Cumulative Reviews book in the print version.

The cumulative reviews are optional; use them as needed. They are named indicating which chapters of the main curriculum the problems in the review come from. For example, "Cumulative Review, Chapter 4" includes problems that cover topics from chapters 1-4.

Both the mixed and cumulative reviews allow you to spot areas that the student has not grasped well or has forgotten. When you find such a topic or concept, you have several options:

1. Check if the worksheet maker lets you make worksheets for that topic.
2. Check for any online games and resources in the Introduction part of the particular chapter in which this topic or concept was taught.
3. If you have the digital version, you could simply reprint the lesson from the student worktext, and have the student restudy that.
4. Perhaps you only assigned 1/2 or 2/3 of the exercise sets in the student book at first, and can now use the remaining exercises.
5. Check if our online practice area at https://www.mathmammoth.com/practice/ has something for that topic.
6. Khan Academy has free online exercises, articles, and videos for most any math topic imaginable.

Concerning challenging word problems and puzzles

While this is not absolutely necessary, I heartily recommend supplementing Math Mammoth with challenging word problems and puzzles. You could do that once a month, for example, or more often if the student enjoys it.

The goal of challenging story problems and puzzles is to **develop the student's logical and abstract thinking and mental discipline**. I recommend starting these in fourth grade, at the latest. Then, students are able to read the problems on their own and have developed mathematical knowledge in many different areas. Of course I am not discouraging students from doing such in earlier grades, either.

Math Mammoth curriculum contains lots of word problems, and they are usually multi-step problems. Several of the lessons utilize a bar model for solving problems. Even so, the problems I have created are usually tied to a specific concept or concepts. I feel students can benefit from solving problems and puzzles that require them to think "out of the box" or are just different from the ones I have written.

I recommend you use the free Math Stars problem-solving newsletters as one of the main resources for puzzles and challenging problems:

Math Stars Problem Solving Newsletter (grades 1-8)
https://www.homeschoolmath.net/teaching/math-stars.php

I have also compiled a list of other resources for problem solving practice, which you can access at this link:

https://l.mathmammoth.com/challengingproblems

Another idea: you can find puzzles online by searching for "brain puzzles for kids," "logic puzzles for kids" or "brain teasers for kids."

Frequently asked questions and contacting us

If you have more questions, please first check the FAQ at https://www.mathmammoth.com/faq-lightblue

If the FAQ does not cover your question, you can then contact us using the contact form at the Math Mammoth.com website.

Chapter 1: The Language of Algebra
Introduction

In the first chapter of *Math Mammoth Grade 7* we both review basic algebra topics from sixth grade and also go deeper into them, plus study the basic properties of the four operations. Since a good part of this chapter is review, it serves as a gentle introduction to 7th grade math, laying a foundation for the rest of the year. For example, when we study integers in the next chapter, students will once again simplify expressions, just with negative numbers. When we study equations in chapters 3 and 5, and also in subsequent grade levels, students will use the skills from this chapter (such as simplifying expressions, using the distributive property) in solving equations.

The main topics are the order of operations, writing and simplifying expressions, and the properties of the four operations, including the distributive property. Students have studied most of these in 6th grade. The main principles are explained and practiced both with visual models and in abstract form, and the lessons contain varying practice problems that approach the concepts from various angles.

Please note that it is not recommended to assign all the exercises by default. Use your judgment, and strive to vary the number of assigned exercises according to the student's needs. See the user guide at the beginning of this book or at **https://www.mathmammoth.com/userguides/** for some further thoughts on using and pacing the curriculum.

You can find matching videos for topics in this chapter at **https://www.mathmammoth.com/videos/** (choose grade 7).

Good Mathematical Practices

- The student is embarking on a wonderful journey into algebra — learning to do arithmetic with letters. The familiar properties of the four operations still hold, just like they do with numbers. Algebra is such a wonderful tool because it allows us to abstract a given situation and represent it symbolically, and then manipulate the representing symbols as if they have a life of their own. It is the foundational tool that allows us to model real-world situations with mathematics.

Pacing Suggestion for Chapter 1

This table does not include the chapter test as it is found in a different book (or file).
Please add one day to the pacing for the test if you use it.

The Lessons in Chapter 1	page	span	suggested pacing	your pacing
Exponents and the Order of Operations	13	*4 pages*	1 day	
Expressions and Equations	17	*3 pages*	1 day	
Properties of the Four Operations	20	*4 pages*	1 day	
Simplifying Expressions	24	*4 pages*	1 day	
Growing Patterns 1	28	*3 pages*	1 day	
The Distributive Property	31	*5 pages*	2 days	
Chapter 1 Review	36	*2 pages*	1 day	
Chapter 1 Test (optional)				
TOTALS		*25 pages*	8 days	

Games at Math Mammoth Online Practice

Hexingo Game — Order of Operations
Practice the order of operations with the four basic operations, parentheses, and exponents.
https://www.mathmammoth.com/practice/order-operations#num=3&operations=add,sub,mult,div,exponents,parens

Expression Exchange
This online activity includes THREE separate work areas where you can explore how simple algebraic expressions work, and then one game. In the work areas, you can learn how to add and subtract simple algebraic terms in order to form an expression. In the game, you will go through practice exercises, forming the asked expressions from parts.
https://www.mathmammoth.com/practice/expression-exchange

Helpful Resources on the Internet

We have compiled a list of Internet resources that match the topics in this chapter, including pages that offer:

- **online practice** for concepts;
- online **games**, or occasionally, printable games;
- **animations** and interactive **illustrations** of math concepts;
- **articles** that teach a math concept.

We heartily recommend you take a look! Many of our customers love using these resources to supplement the bookwork. You can use these resources as you see fit for extra practice, to illustrate a concept better and even just for some fun. Enjoy!

https://l.mathmammoth.com/gr7ch1

SCAN ME

Exponents and the Order of Operations

Let's review! Exponents are a shorthand for writing repeated multiplications by the same number.

For example, 0.9 · 0.9 · 0.9 · 0.9 · 0.9 is written 0.9^5.

The tiny raised number is called the **exponent**.
It tells us how many times the **base** number is multiplied by itself.

exponent

$$12^4 = 12 \times 12 \times 12 \times 12$$
$$= 20{,}736$$

base

The expression 2^5 is read as "two to the fifth power," "two to the fifth," or "two raised to the fifth power."

Similarly, 0.7^8 is read as "seven tenths to the eighth power" or "zero point seven to the eighth."

The "powers of 6" are simply expressions where 6 is raised to some power: for example, 6^3, 6^4, 6^{45}, and 6^{99} are powers of 6.

Expressions with the exponent 2 are usually read as something "**squared.**" For example, 11^2 is read as "eleven squared." That is because it gives us the area of a square with sides 11 units long.

Similarly, if the exponent is 3, the expression is usually read using the word "**cubed.**" For example, 1.5^3 is read as "one point five cubed" because it is the volume of a cube with an edge 1.5 units long.

1. Evaluate.

a. 4^3 **b.** 10^5 **c.** 0.1^2

d. 0.2^3 **e.** 1^{100} **f.** 100 cubed

2. Write these expressions using exponents. Find their values.

a. 0 · 0 · 0 · 0 · 0 **b.** 0.9 · 0.9

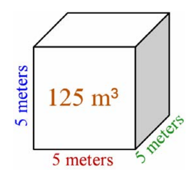
c. 5 · 5 · 5 + 2 · 2 · 2 · 2 · 2

d. 6 · 10 · 10 · 10 · 10 · 10 · 10 − 9 · 10 · 10 · 10 · 10

The expression $(5\text{ m})^3$ means that we multiply 5 meters by itself three times:

$(5\text{ m})^3 = 5\text{ m} \cdot 5\text{ m} \cdot 5\text{ m} = 125\text{ m}^3$

Notice that $(5\text{ m})^3$ is different from 5 m^3. The latter has no parentheses, so the exponent (the little 3) applies only to the unit "m" and not to the whole quantity 5 m.



3. Find the value of the expressions. Include the proper unit.

a. $(2\text{ cm})^3$ **b.** $(11\text{ ft})^2$ **c.** $(1.2\text{ km})^2$ **d.** $(6\text{ in})^2$

4. Match each of (a) and (b) with one expression on the right.

a. The volume of a cube with edges 2 cm long.

b. The volume of a cube with edges 8 cm long.

(i) 8 cm^3 **(ii)** $(8\text{ cm})^3$ **(iii)** 512 cm

The Order of Operations — PE[MD][AS]

1) Solve what is within parentheses (**P**).

2) Solve exponents (**E**).

3) Solve the multiplicative operations — this includes both multiplications (**M**) and divisions (**D**) — from left to right.

4) Solve the additive operations — this includes both additions (**A**) and subtractions (**S**) — from left to right.

Example 1. In $15 - 2 + 3 \cdot 3$, we do $3 \cdot 3$ first, then the subtraction, and lastly the addition.

You can remember PEMDAS with the silly mnemonic *Please Excuse My Dear Aunt Sally*. Or make up your own!

5. Find the value of each expression.

a. $120 - (9 - 4)^2$ **b.** $120 - 9 - 4^2$	**c.** $4 \cdot 5^2$ **d.** $(4 \cdot 5)^2$	**e.** $10 \cdot 2^3 \cdot 5^2$ **f.** $10 + 2^3 \cdot 5^2$
g. $(0.2 + 0.3)^2 \cdot (5 - 5)^4$	**h.** $0.7 \cdot (1 - 0.3)^2$	**i.** $20 + (2 \cdot 6 + 3)^2$

Example 2. Simplify $(10 - (5 - 2))^2$.

Here we have double parentheses. First calculate what is within the *inner* parentheses: $5 - 2 = 3$. Then the expression becomes $(10 - 3)^2$.

The rest is easy:
$(10 - 3)^2 = 7^2 = 49$.

Example 3. Simplify $2 + \dfrac{1 + 5}{40 - 6^2}$.

The fraction line works just like parentheses, as a grouping symbol, grouping both what is above the line and also what is below it. Therefore, first solve what is in the numerator and in the denominator (in either order).

$$2 + \frac{1 + 5}{40 - 6^2} = 2 + \frac{6}{4} = 2 + \frac{2}{3} = \frac{8}{3}$$

6. Find the value of each expression.

a. $(12 - (9 - 4)) \cdot 5$	**b.** $12 - (9 - (4 + 2))$	**c.** $(10 - (8 - 5))^2$	**d.** $3 \cdot (2 - (1 - 0.4))$

7. Find the value of each expression.

a. $\dfrac{4 \cdot 5}{2} \cdot \dfrac{9}{3}$	**b.** $\dfrac{4 \cdot 5}{2} + \dfrac{9}{3}$	**c.** $\dfrac{4 + 5}{2} + \dfrac{9}{3 - 1}$

In algebra and beyond, the fraction line is preferred over the ÷ symbol, and it acts as a grouping symbol (just like parentheses).

Compare how each of these expressions looks when written either with the division symbol or with the fraction line. The latter usually makes the expressions easier to read.

$46 \div 2 + 50 \div 5$ vs. $\dfrac{46}{2} + \dfrac{50}{5}$

$48 \div (1 + 5) \cdot 3$ vs. $\dfrac{48}{1+5} \cdot 3$

Notice how only what comes directly after the ÷ symbol, whether a single number or an expression in parentheses, goes to the denominator.

Example 4. Rewrite the expression (10 + 8) ÷ 4 + 3 using the fraction line.

The denominator is just 4, not 4 + 3. The 10 + 8 will not need parentheses anymore because the fraction line in itself is a grouping symbol. So, this is written as $\dfrac{10+8}{4} + 3$.

The additions and subtractions that are done last (*not* additions and subtractions in parentheses or in the numerator/denominator) separate the expression into subsections that we call *terms*.

Example 5. This expression has four terms, separated by a +, then a −, and lastly a + sign.

$3^2 + \dfrac{2}{4} - \dfrac{30}{6+2} + 4 \cdot 8$

Example 6. Rewrite the expression 2 ÷ 4 + 3 ÷ (7 + 2) using the fraction line.

Now there are *two* divisions: the first by 4 and the second by (7 + 2), separated by an addition. This means we will use two fractions, or two terms, in the expression. It is written as $\dfrac{2}{4} + \dfrac{3}{7+2}$.

8. Match the expressions that are the same.

$2 \div 3 \cdot 4$ | $2 \div (3 \cdot 4)$ | $1 + 3 \div (4 + 2)$ | $1 + 3 \div 4 + 2$ | $(1 + 3) \div 4 + 2$ | $(1 + 3) \div (4 + 2)$

$1 + \dfrac{3}{4} + 2$ | $\dfrac{1+3}{4+2}$ | $\dfrac{2}{3} \cdot 4$ | $\dfrac{2}{3 \cdot 4}$ | $\dfrac{1+3}{4} + 2$ | $1 + \dfrac{3}{4+2}$

9. Rewrite each expression using the fraction line and then find its value.

a. 56 ÷ 7 + 6	**b.** 7 ÷ (2 + 6)	**c.** 16 ÷ (2 + 6) − 2	**d.** 4 ÷ 5 − 1 ÷ 3

To **evaluate an expression** means to find (calculate) its value.

Example 7. Evaluate the expression $x^2 - \frac{2+y}{y}$ when x is 10 and y is 3.

This means we substitute 10 for x and 3 for y in the expression and then calculate its value according to the order of operations:

$$x^2 - \frac{2+y}{y} = 10^2 - \frac{2+3}{3} = 100 - \frac{5}{3} = 98\ 1/3$$

However, in algebra and beyond, it is customary to *not* give answers as mixed numbers but as fractions, to avoid confusion. After all, 98 1/3 could easily be mistaken for 981/3. So let’s go back to the expression 100 − (5/3) and simplify it so it becomes a fraction:

$$100 - \frac{5}{3} = \frac{300}{3} - \frac{5}{3} = \frac{295}{3}$$ (This is the final value as a fraction.)

10. Find the value of these expressions. (When applicable, give your answer as a fraction, not as a decimal.)

a. $\frac{9^2}{9} \cdot 6$	**b.** $\frac{2^3}{3^2}$	**c.** $\frac{(5-3)\cdot 2}{8-1+2} + 3$

11. Evaluate the expressions. (When applicable, give your answer as a fraction, not as a decimal.)

a. $2x^2 - x$, when $x = 4$	**b.** $\frac{3s}{5} - \frac{2t}{5}$, when $s = 10$ and $t = 4$
c. $\frac{x^2}{x+1}$, when $x = 3$	**d.** $\frac{a+b}{b} + 2$, when $a = 1$ and $b = 3$

12. Why is it wrong to write the expression $2 + 5 \cdot 2 \div 4$ as $\frac{2+5\cdot 2}{4}$?

Expressions and Equations

Expressions in mathematics consist of:

- numbers;
- mathematical operations (+ , − , · , ÷ , exponents);
- and letter variables, such as x, y, a, T, and so on.

Note: Expressions do *not* have an "equals" sign!

Examples of expressions: 5 $\quad \frac{xy^4}{2}$ $\quad T - 5 + \frac{x}{7}$

An **equation** has two expressions separated by an equals sign:

(expression 1) = (expression 2)

Examples: $0 = 0$ $\quad 2(a - 6) = b$

$9 = -8$ (a false equation) $\quad \frac{x + 3}{2} = 1.5$

What do we do with expressions?

We can find the *value* of an expression (*evaluate* it). If the expression contains variables, we cannot find its value unless we know the value of the variables.

For example, to find the value of the expression $2x$ when x is 6/7, we simply substitute 6/7 in place of x.
We get $2x = 2 \cdot (6/7) = 12/7$.

Note: When we write $2x = 2 \cdot (6/7) = 12/7$, the equals sign is *not* signaling an equation to solve. (In fact, we already know the value of x!) It is simply used to show that the value of the expression $2x$ here is the same as the value of $2 \cdot (6/7)$, which is in turn the same as 12/7.

What do we do with equations?

If the equation has a variable (or several) in it, we can try to *solve* the equation. This means we find the values of the variable(s) that make the equation true.

For example, we can solve the equation $0.5 + x = 1.1$ for the unknown x.

The value 0.6 makes the equation true: $0.5 + 0.6 = 1.1$. We say $x = 0.6$ is the **solution** or the **root** of the equation.

1. This is a review. Write an expression.

a. $2x$ minus the sum of 40 and x.

b. The quantity 3 times x, cubed.

c. s decreased by 6

d. five times b to the fifth power

e. seven times the quantity x minus y

f. the difference of t squared and s squared

g. x less than 2 cubed

h. the quotient of 5 and y squared

i. 2 less than x to the fifth power

j. x cubed times y squared

k. the quantity $2x$ plus 1 to the fourth power

l. the quantity x minus y divided by the quantity x squared plus one

To read the expression $2(x + y)$, use the word ***quantity***: "two times the quantity x plus y."

There are other ways, as well, just not as common:

"two times the sum of x and y," or "the product of 2 and the sum x plus y."

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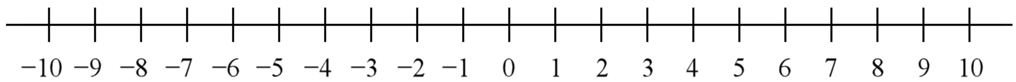
Integers



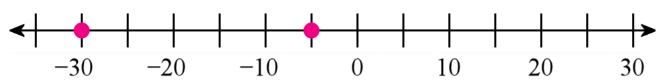
The **counting numbers** are 1, 2, 3, 4, 5, and so on. They work for addition. But counting numbers do not allow us to perform all possible subtractions; for example, the answer to the problem 2 − 7 is not any of them. That is when we come to the *negatives* of the counting numbers: −1, −2, −3, −4, −5, and so on.

Together with zero, all these form the set of **integers**: {..., −4, −3, −2, −1, 0, 1, 2, 3, 4, ...}.
Note: Zero is neither positive nor negative.

Read −1 as "negative one" and −5 as "negative five." Some people read −5 as "minus five." That is very common, and it is not wrong, but be sure that you do not confuse it with subtraction.

Often, we need to put parentheses around negative numbers in order to avoid confusion with other symbols. Therefore, $^{-}5$, −5, and (−5) all mean "negative five."

Which is more, −30 or −5?



Which is *warmer*, −30°C or −5°C? Clearly −5°C is. Temperatures get colder and colder the more they move towards the negative numbers. We can write a comparison: −30°C < −5°C.

Similarly, we can write −$500 < −$200 to signify that to owe $500 is a worse situation than to owe $200.

Or, in elevation, −15 m > − 50 m means that 15 m below sea level is higher than 50 m below sea level.

1. Write comparisons using > , < , and integers. Don't forget to include the units!

a. The temperature at 5 AM was 12°C below zero. Now, at 9 AM, it is 8°C below zero.

b. I owe my mom $12, and my sister owes her $25.

c. The bottom of the Challenger Deep trench is 11,033 m below sea level.
Mt. Everest reaches to a height of 8,848 m.

d. The total electric charge of five electrons is −5*e*. The total electric charge of 5 protons is +5*e*.
(The symbol *e* means elementary charge, or a charge of a single proton.)

e. Dean has $55, whereas Jack owes $15.

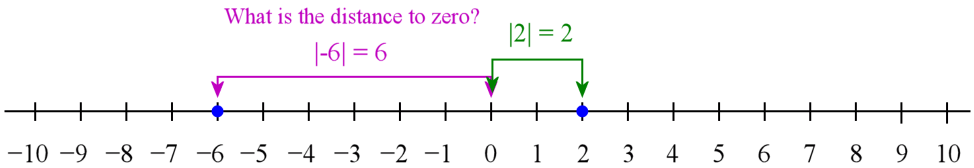
2. Which integer is...

a. 3 more than −7 **b.** 8 more than −3 **c.** 7 less than 2 **d.** 5 less than −11

3. Write the numbers in order from the least to the greatest.

a. −5 −56 51 −15	**b.** 3 0 −31 −13

The **absolute value** of a number is its distance to zero.



We denote the absolute value of a number by putting vertical bars on either side of it.
So $|-4|$ means "the absolute value of 4," which is 4. Similarly, $|87| = 87$. In an expression we treat the absolute value bars like parentheses and solve them first.

Example 1. Simplify $|-4| - |1|$. First simplify the absolute values. We get $4 - 1 = 3$.

Let's say someone's account balance is −$1,000. That person is in debt. The absolute value of the debt is written as |−$1000| and means that the *size* of the debt is $1,000.

If a diver is at a depth of −22 m, the absolute value $|-22 \text{ m}|$ tells us how far he is from the surface (22 m).

4. Simplify.

a. $|-11|$	**b.** $|+7|$	**c.** $|0|$	**d.** $|-46|$
e. $|-5| + |-2|$		**f.** $|-5| - |2|$	
g. $|-5| + |-2| + |8|$		**h.** $|5| + |-2| - |-1|$	

5. Answer, using the absolute value notation.

a. What is the distance between −153 and zero on a number line?

b. What is the distance between x and zero on a number line?

6. Interpret the absolute value in each situation.

a. A fishing net is at the depth of 15 feet. $|-15 \text{ ft}|$ = ________ft

Here, the absolute value shows ____

b. The temperature is −5°C. |−5°C | = ________°C

Here, the absolute value shows ____

c. Eric's balance is −$7. |−$7| = $________

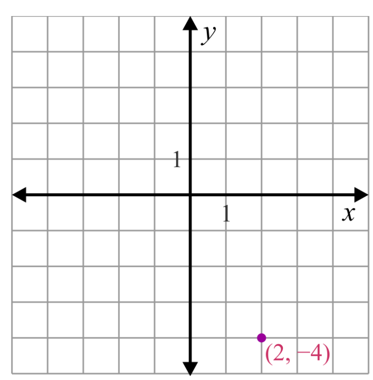
The absolute value shows ____________________________

____

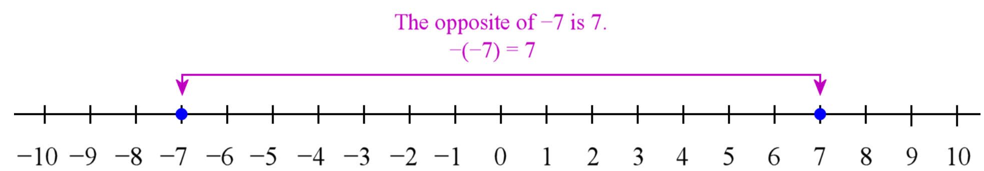
d. A point is drawn in the coordinate grid at (2, −4). $|-4|$ = ________

Here, the absolute value shows ________________________

____



The **opposite** of a number is the number that is on the opposite side of zero at the same distance from zero.



We denote the opposite of a number using the minus sign. For example, the opposite of 4 is written as -4. The opposite of -2 is written as $-(-2)$, which is of course 2. So, $-(-2) = 2$.

The opposite of zero is zero itself. In symbols, $-0 = 0$.

"But wait," you might ask, "doesn't -4 mean 'negative four,' not 'the opposite of four'?"

It can mean either! Sometimes the context will help you tell which is which. Other times it isn't necessary to differentiate, because, after all, the opposite of four *is* negative four, or $-4 = -4$.

In the expression $-(4 + 5)$, the minus sign means the opposite of the sum $4 + 5$, which equals negative nine.

Example 2. $-|7|$ means the opposite of the absolute value of seven. It simplifies to -7.

Notice that there are *three* different meanings for the minus sign:

1. To indicate subtraction, as in $7 - 2$.
2. To indicate negative numbers: "negative 7" is written -7.
3. To indicate the opposite of a number: $-(n + 1)$ is the opposite of $n + 1$.

7. Write using symbols, and simplify if possible.

a. the opposite of 6

b. the opposite of -11

c. the opposite of the absolute value of 12

d. the absolute value of negative 12

e. the opposite of the sum $6 + 8$

f. the opposite of the difference $9 - 7$

g. the absolute value of the opposite of 8

h. the absolute value of the opposite of -2

8. Simplify.

a. $-|8|$ **b.** $-(-9)$ **c.** $-|-7|$ **d.** -0 **e.** $-(-(-100))$

9. Write with symbols. Use a variable for "*a number*" or "*a certain number*".

a. The absolute value of a number is equal to 6.

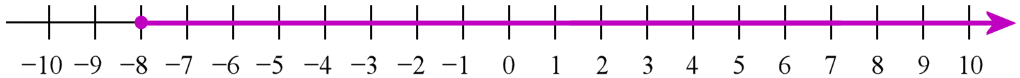
b. The opposite of a certain number is less than negative 2.

c. The absolute value of a certain number is greater than 15.

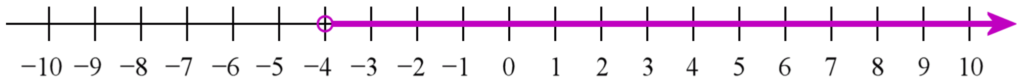
d. The opposite of n is equal to to the sum $56 + 5$.

10. Daniel owed \$5. Then he borrowed \$10 more. Next, he paid off \$7 of his debts. Lastly he made yet another debt of \$4. Write one integer to express Daniel's money situation now.

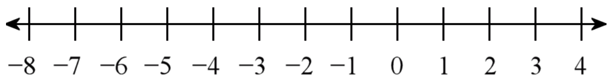
Remember **inequalities?** The number line below illustrates the inequality $x \geq -8$. Notice the arrow on the right, which shows that the ray continues to infinity. The closed circle denotes that −8 belongs to the solution set.



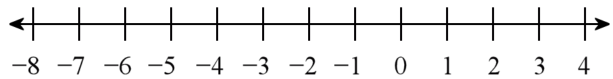
The inequality $x > -4$ is plotted on the number line below. The open circle indicates that −4 is not part of the solution set.



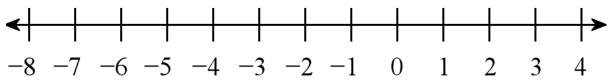
11. Plot these inequalities on the number line. Don't forget the arrow on the open end.



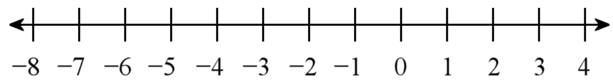
a. $x < -3$



b. $x > -1$



c. $x \geq -2$



d. $x \leq 2$

12. **a.** Solve the inequality $x < 2$ in the set {−3, −2, −1, 0, 1, 2, 3}.

b. Solve the inequality $x \geq -5$ in the set {−10, −8, −6, −2, −1, 0, 5}.

13. Write an inequality. Use negative integers where appropriate.

a. The pit is at most 10 m deep.

b. The pit is at least 12 m deep.

c. Tim's debt is no more than $500.

d. Nora owes at least $100.

e. For the skiing contest to take place, the temperature has to be warmer than 15 degrees below zero.

f. The freezer temperature should be colder than 10 degrees below zero.

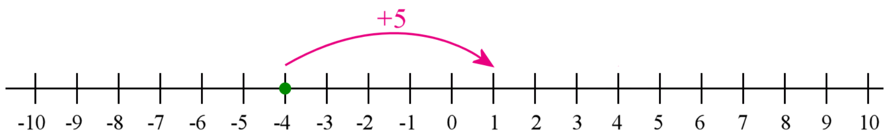
Puzzle Corner

Let a and b be two negative integers, with $b > a$.
What is the distance between them on the number line?
Write an expression.

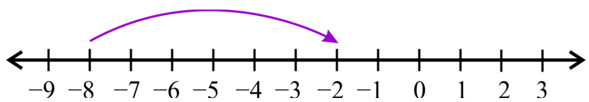
Addition and Subtraction on the Number Line 1

Addition can be modeled on the number line as a movement to the *right*.

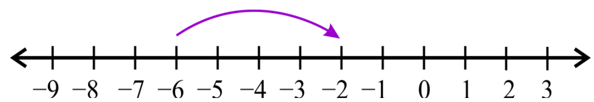
Suppose you are at −4, and you jump 5 steps to the right. You end up at 1. We write the addition −4 + 5 = 1.



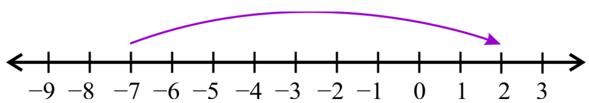
1. Write an addition equation to match each number line jump.



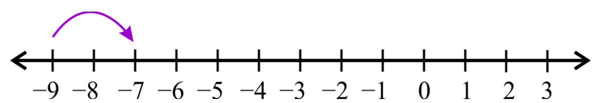
a. _______ + _______ = _______



b. _______ + _______ = _______

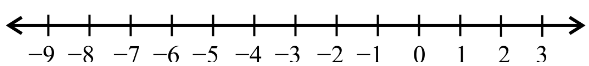


c. _______ + _______ = _______

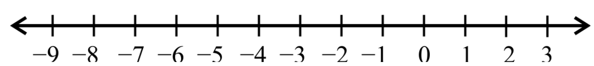


d. _______ + _______ = _______

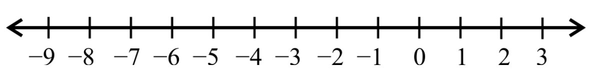
2. Draw a number line jump for each addition equation and solve.



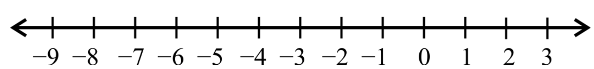
a. −8 + 3 = _______



b. −2 + 5 = _______



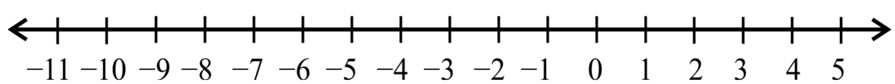
c. −4 + 4 = _______



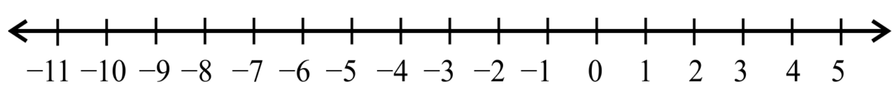
d. −10 + 12 = _______

3. What about adding more than one number? How could these additions be illustrated by number line jumps?

a. −4 + 2 + 3



b. −11 + 6 + 4



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Solving Equations

Do you remember? An **equation** has two expressions, separated by an equals sign:

(expression) = **(expression)**

To solve an equation, we can

- add the same quantity to both sides
- subtract the same quantity from both sides
- multiply both sides by the same number
- divide both sides by the same number

Notice that in any of these operations, the two expressions on the left and right sides of the equation will remain equal, even though the expressions themselves change!

Example 1. We will manipulate the simple equation 2 + 3 = 5 in these four ways. We will write in the margin the operation that is going to be done next to both sides.

Let's add six to both sides.	$2 + 3 = 5$	**+ 6**
Now, both sides equal 11. Next, we multiply both sides by 8.	$2 + 3 + 6 = 11$	**· 8**
Now, both sides equal 88. Next, we subtract 12 from both sides.	$16 + 24 + 48 = 88$	**− 12**
Now both sides equal 76. Next, we divide both sides by 2.	$16 + 24 + 48 - 12 = 76$	**÷ 2**
Now both sides equal 38.	$8 + 12 + 24 - 6 = 38$	

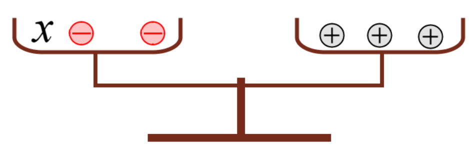
Of course, you do not usually work with equations like the one above, but with ones that have an unknown. Your goal is to **isolate** the unknown, or **leave it by itself,** on one side. Then the equation is solved.

We can model an equation with a **pan balance**. Both sides (pans) of the balance will have an *equal* weight in them, thus the sides are balanced (not tipped to either side).

Example 2. Solve the equation $x - 2 = 3$.

We can write this equation as $x + (-2) = 3$ and model it using negative and positive counters in the balance.

Here x is accompanied by two negatives on the left side. Adding two <u>positives</u> *to both sides* will cancel those two negatives. We denote that by writing "**+2**" in the margin.

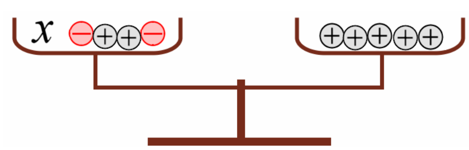


margin ↓

$x + (-2) = 3$ | **+ 2**

↓

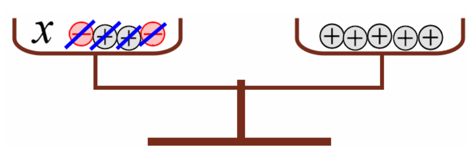
We write $x + (-2) + 2 = 3 + 2$ to show that 2 was added to both sides of the equation.



$x + (-2)\ + 2 = 3\ + 2$

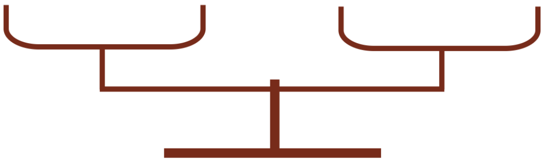
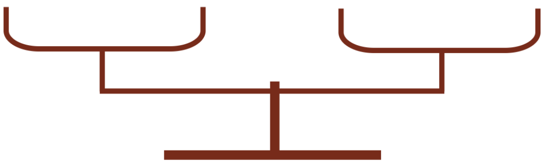
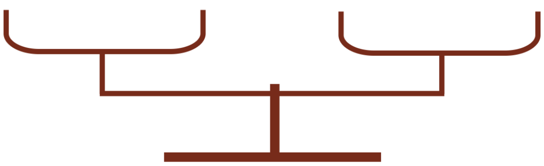
↓

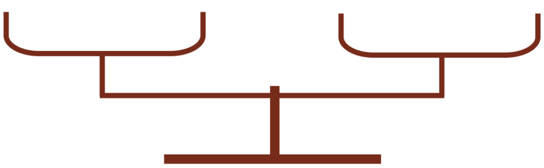
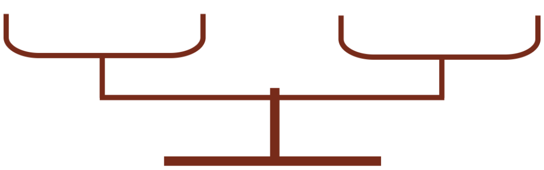
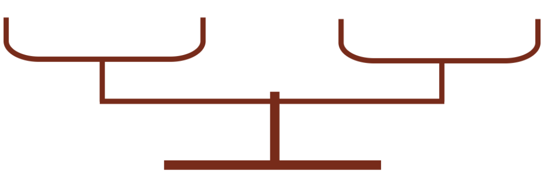
Now the two positives and two negatives on the left side cancel each other, and x is left by itself. On the right side we have 5, so x equals 5 positives.

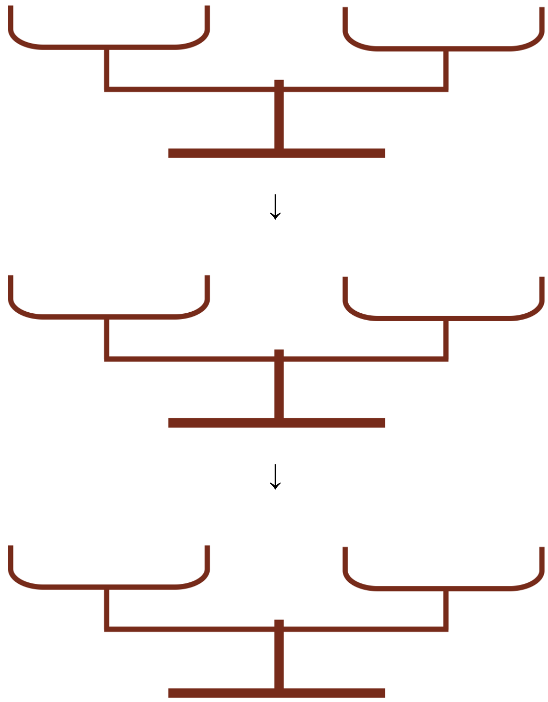


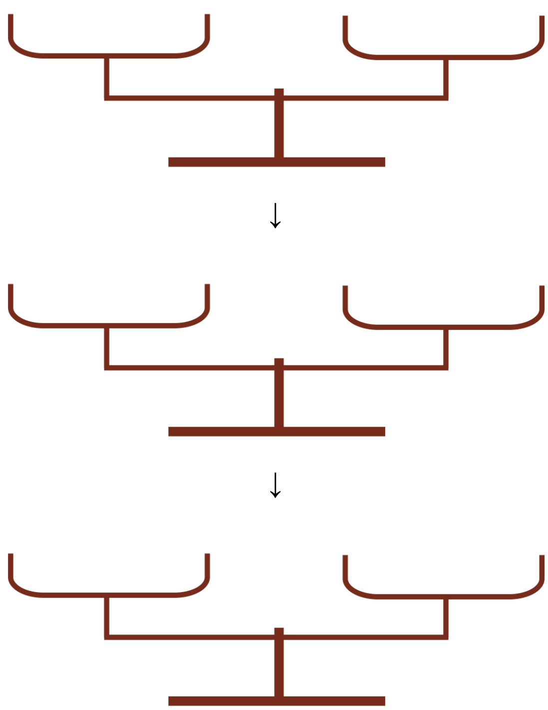
$x = 5$

1. Solve the equations. Write in the margin what operation you do to both sides.

a. Balance	Equation	Operation to do to both sides
 ↓  ↓ 	$x + 1 = -4$	

b. Balance	Equation	Operation to do to both sides
 ↓  ↓ 	$x - 1 = -3$	

c. Balance	Equation	Operation
	$x - 2 = 6$	

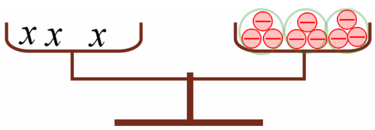
d. Balance	Equation	Operation
	$x + 5 = 2$	

2. If you need more practice, solve the following equations also. Draw a balance in your notebook to help you.

a. $x + (-3) = 7$ **b.** $x + (-3) = -4$ **c.** $x + 6 = -1$ **d.** $x + 5 = -4$

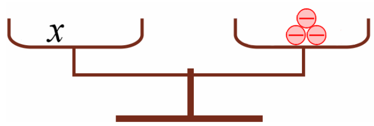
In the examples below, we either multiply or divide both sides by the same number. Study them carefully!

Example 3.



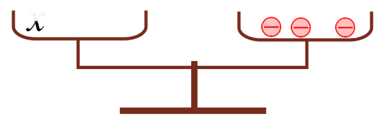
$3x = -9$ | ÷ 3

↓



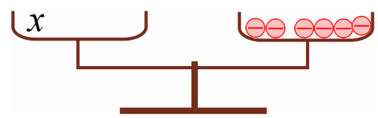
$x = -3$

Example 4.



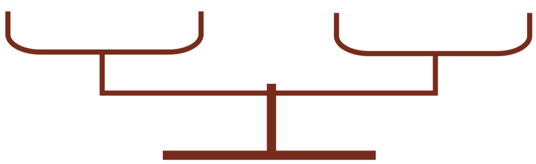
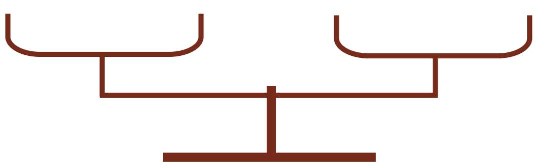
$(1/2)x = -3$ | · 2

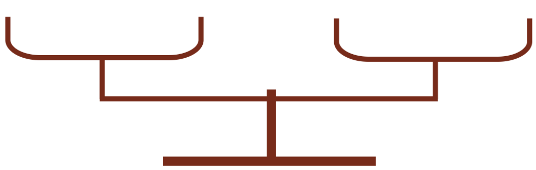
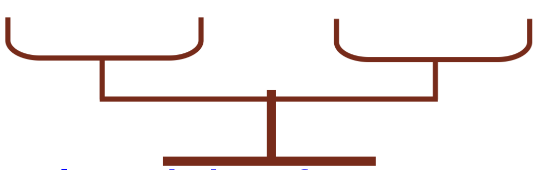
↓



$x = -6$

3. Solve the equations. Write in the margin what operation you do to both sides.

a. Balance	Equation	Operation to do to both sides
 ↓ 	$4x = -12$	

b. Balance	Equation	Operation to do to both sides
 ↓ 	$(1/3)x = -5$	

4. Let's review a little! Which equation matches the situation?

a. A stuffed lion costs $8 less than a stuffed elephant.
Note: p_l signifies the price of the lion and p_e the price of the elephant.

$p_e = 8 - p_l$	$p_e = p_l - 8$	$p_l = p_e - 8$

b. A shirt is discounted by 1/5 of its price, and now it costs $16.

$p - 1/5 = \$16$	$\dfrac{4p}{5} = \$16$	$\dfrac{p}{5} = \$16$	$p - p/5 = \$16$	$\dfrac{5p}{4} = \$16$

5. Find the roots of the equation $\dfrac{6}{x+1} = -3$ in the set $\{2, -2, 3, -3, 4, -4\}$.

6. Write an equation, then solve it using guess and check. Each root is between −20 and 20.

a. 7 less than x equals 5.

b. 5 minus 8 equals x plus 1

c. The quantity x minus 1 divided by 2 is equal to 4.

d. x cubed equals 8

e. −3 is equal to the quotient of 15 and y

f. Five times the quantity x plus 1 equals 10.

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Chapter 3 Mixed Review

1. Write an expression.
(Expressions and Equations/Ch1)

a. 10 less than x squared.

b. The quotient of 154 and k cubed.

c. The quantity x plus 2 to the fifth power.

d. x plus 2 to the fifth power.

2. The sides of a square are (x + 2) long. (Simplifying Expressions/Ch1)

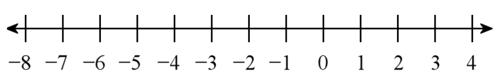
a. Sketch the square.

b. Write an expression for the area of the square.

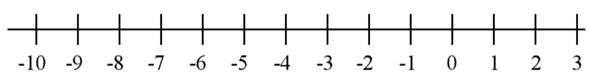
c. Write an expression for the perimeter of the square.

d. Evaluate your expression for the area of the square when x = 1.5.

3. Draw a number line jump for each addition or subtraction. (Addition and Subtraction on the Number Line 1/Ch2)



a. −2 + 6 = ______



b. −3 − 5 = ______

4. Draw counters for the addition 3 + (−5). Explain how to perform the addition using the counters.
(More Addition of Integers/Ch2)

5. Solve. (More Addition of Integers/Ch2)

a. 89 + (−35) =	**b.** −45 + (−29) =	**c.** −78 + 60 =

6. Change each addition into a subtraction or vice versa. Then solve whichever is easier. (Subtraction of Integers/Ch2)

a. −2 + (−18) ↓ _____ − _____ = _______	**b.** 56 − (−34) ↓ _____ + _____ = _______	**c.** −14 + (−24) ↓ _____ − _____ = _______	**d.** 2 + 9 ↓ _____ − _____ = _______

7. Write comparisons using > , < , and integers. Include the units, too. (Integers/Ch2)

a. The temperature at the North Pole is −34 degrees Celsius, whereas in New York, it is −8 degrees Celsius.

b. The total electric charge of 12 electrons is $-12e$. The total electric charge of 3 protons is $+3e$.

8. Name the property of arithmetic illustrated by the equation $2x = x \cdot 2$.
(Properties of the Four Operations/Ch1)

9. Evaluate the expression $|a - b|$ for the given values of a and b. Check that the answer you get is the same as if you had used a number line to figure out the distance between the two numbers.
(Distance Between Numbers/Ch2)

a. a is 8 and b is 54	**b.** a is −12 and b is −5

10. Describe a situation where one person has a positive account balance and another has a negative balance, and the one person's balance is \$30 more. (Integers/Ch2)

11. Use the distributive property "backwards" to write the expression as a product. (The Distributive Property/Ch1)

a. $42s + 28$ = _____ (_____ + _____)

b. $54z - 18$ = _____ (_____ − _____)

12. Find the missing numbers. (Various lessons/Ch2)

a. ________ ÷ (−5) = 35	**b.** 35 ÷ ________ = −5	**c.** 5 · _______ = −35
d. 2 + (−5) = _______ + 7	**e.** 2 · 9 = −3 · ________	**f.** 40 ÷ _______ = −5 · 4

13. Write the equation and then solve it using "guess and check." Each root is between −20 and 20.
(Expressions and Equations/Ch1)

a. 2 plus 14 equals x minus 1
b. x cubed equals 27

14. Add or subtract. (Various lessons/Ch2)

a. $(-9) + (-18) =$ _______	**b.** $-21 - (-3) =$ _______	**c.** $17 - 51 =$ _______

15. Give a real-life situation for the sum $3 + (-10)$. (Various lessons/Ch2)

16. Simplify. (Integers/Ch2)

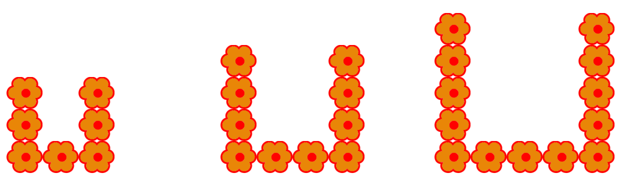
a. $|-2|$ **b.** $-(-2)$ **c.** $-|2|$ **d.** -0

17. Find the value of the expressions when $x = -2$ and $y = 8$. (Multiplying Integers 2/Ch2)

a. $5x^2$	**b.** $-5y + 6$	**c.** $-(y + x)$

18. Jeremy is 2 years older than Larry. Write an expression for Larry's age, if Jeremy is y years old.
(Expressions and Equations/Ch1)

19. Here is a growing pattern. Draw the steps 4 and 5 and answer the questions. (Growing Patterns 1/Ch1)



Step 1 2 3

a. How do you see this pattern grow?

b. How many flowers will be in step 39?

c. In step n?

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Rational Numbers

If you can write a number as a *ratio of two integers*, it is a **rational number.**

For example, 4.3 is a rational number because we can write it as the ratio $\frac{43}{10}$ or 43:10.

Note: To represent rational numbers, we usually indicate the ratio with a fraction line rather than a colon.

Examples of rational numbers

Since −10 can be written as $\frac{-10}{1}$, it is a rational number. It can also be written as $\frac{10}{-1}$.

Since 0.1 can be written as $\frac{1}{10}$, it is a rational number.

Since 3.24 can be written as $\frac{324}{100}$, it, too, is a rational number.

Negative fractions

The ratio of the integers 7 and −10 gives us the fraction $\frac{7}{-10}$. As we studied earlier, we usually write this as $-\frac{7}{10}$ and read it as "negative seven tenths."

Obviously, all fractions, whether negative or positive, are rational numbers.

Negative fractions give us negative decimals.

For example, $-\frac{8}{10}$ is written as a decimal as −0.8, and $-5\frac{21}{100} = -5.21$.

You can write a rational number as a ratio of two integers in many ways.

For example, the decimal −1.4 can be written as a ratio of two integers in all these ways (and more!):

$$-1.4 = \frac{-14}{10} = \frac{-28}{20} = \frac{28}{-20} = \frac{42}{-30} = \frac{-42}{30} = \frac{-7}{5}$$

So −1.4 is *definitely* a rational number! ☺ But the same holds true for all rational numbers—you can always write them as a ratio of two integers in multitudes of ways.

1. Write these numbers as a ratio (fraction) of two integers.

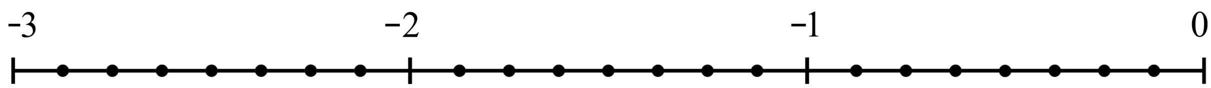
a. 6	**b.** −100	**c.** 0	**d.** 0.21
e. −1.9	**f.** −5.4	**g.** −0.56	**h.** 0.022

2. Are all percents, such as 34% or 5%, rational numbers? Justify your answer.

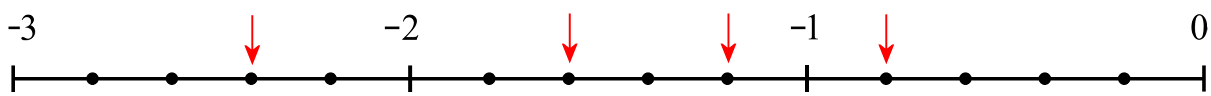
3. Form a fraction from the two given integers. Then convert it into a decimal.

a. 8 and 5	**b.** −4 and 10	**c.** 89 and −100
d. −5 and 2	**e.** 91 and −1000	**f.** −1 and −4

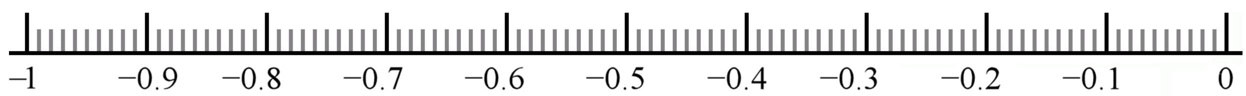
4. Mark the fractions and mixed numbers on the number line below: $-\frac{1}{2}$, $-\frac{7}{8}$, $-1\frac{5}{8}$, $-\frac{9}{4}$, $-2\frac{3}{4}$



5. Write the fractions marked by the arrows.



6. Mark the decimals on the number line: −0.11, −0.58, −0.72, −0.04



7. Sketch a number line from −3 to 0, with tick marks at every tenth. Then mark the following numbers on your number line: −0.2, −1.5, −2.8, −3/5, and −5/2.

8. Write these rational numbers as ratios of two integers (fractions) in a lot of different ways.

a. $-2 \; = \; -\frac{2}{1} \; =$
b. $0.6 \; = \; \frac{6}{10} \; =$

9. Compare, writing < or > in between the numbers.

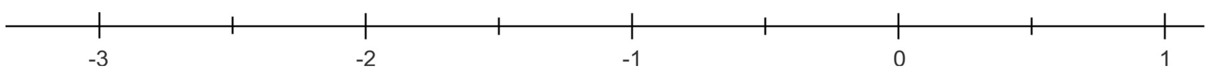
a. $-\frac{7}{8}$ $\quad$ -1	**b.** $-\frac{3}{4}$ $\quad$ $\frac{1}{2}$	**c.** $-\frac{15}{2}$ $\quad$ -7	**d.** -0.98 $\quad$ -1.4

10. Order these rational numbers in order, from the smallest to the greatest.

$2.1 \quad -\frac{1}{8} \quad -1 \quad -\frac{7}{3} \quad -2.01 \quad 1 \quad \frac{1}{3} \quad -0.5$

11. Mark the decimals *and* the fractions on the number line, approximately.

$0.3 \quad -\frac{2}{5} \quad -0.8 \quad -\frac{10}{4} \quad -2.1 \quad -1\frac{1}{2} \quad -\frac{17}{10} \quad 0.95$



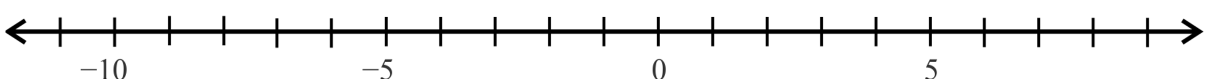
Recall that the absolute value of a number is its distance from zero.

Below, the thickened line shows the set of numbers that are greater than −3 and at the same time, less than 3. We can write it like this: the set of numbers x so that $-3 < x < 3$.

These are the numbers whose absolute value is less than 3, in other words the set of numbers for which $|x| < 3$. Their distance to zero is less than 3. For example, −2.8 and 0.492 and −6/5 belong to this set.

Note that 3 and −3 are not part of this set; that is why we use an open circle at 3 and −3.

12. **a.** Show on the number line the set of numbers x for which $|x| < 1.5$



b. List three rational numbers in that set that are not integers.

13. List three rational numbers r so that $|r| < 2$ and $r > -1$.

Repeating Decimals

As you already know, sometimes it is easy to write a fraction as a decimal. For example, 3/10 = 0.3 and 1/4 = 0.25. However, if you don't know of any other way to to find the decimal equivalent of a fraction, the technique that works all the time is to **treat the fraction as a division** and divide.

Example 1. Write $\frac{31}{40}$ as a decimal.

We will use long division. Note how we add many decimal zeros to the dividend (31) so that we can continue the division into the decimal digits.

This division **terminates** (comes out even) after just three decimal digits.

We get $\frac{31}{40}$ = 0.775. This is a **terminating decimal**.

$$\begin{array}{r} 0\,0.7\,7\,5 \\ 40\overline{)\,3\,1.0\,0\,0\,0} \\ \underline{-2\,8\,0} \\ 3\,0\,0 \\ \underline{-2\,8\,0} \\ 2\,0\,0 \\ \underline{-\,2\,0\,0} \\ 0 \end{array}$$

1. Write as decimals, using long division. Continue the division until it terminates.

a. $\frac{3}{16}$	**b.** $\frac{51}{32}$	**c.** $\frac{17}{80}$

2. Use long division to write these fractions as decimals. Continue the division to at least 6 decimal digits. Notice what happens!

a. $\frac{2}{3}$	**b.** $\frac{7}{11}$	**c.** $\frac{8}{9}$

Example 2. Write $\frac{18}{11}$ as a decimal.

```
      0 1.6 3 6 3
11 ) 1 8.0 0 0 0
    -1 1
       7 0
     - 6 6
         4 0
       - 3 3
           7 0
         - 6 6
             4 0
           - 3 3
               7
```

We write 18 as 18.0000 in the long division "corner" and divide by 11. Notice how the digits "63" in the quotient and the remainders 40 and 70 start repeating.

So $\frac{18}{11}$ = 1.636363... . We can use an ellipsis (three dots, or "…") to indicate that the decimal is non-terminating. A better notation is to draw a **bar** (a line) over the digits that repeat: 1.636363… = $1.\overline{63}$.

This number is called a **repeating decimal** because the digits "63" repeat forever!

The decimal form of ANY rational number is either a terminating decimal or a repeating decimal.

This is an important fact. It says that when you write any fraction as a decimal, there are only two possibilities: either the decimal terminates or it repeats.

The converse is also true: if a decimal terminates or is a repeating decimal, it *can* be written as a fraction, thus is a rational number.

Example 3. The repeating decimal 1.9051050505… is written as $1.9051\overline{05}$. Notice that the bar marks only the digits that repeat ("05"). The digits "9051" that don't repeat are not included under the bar.
(If you're curious, as a fraction, this number is 1,886,054/990,000.)

Example 4. A calculator gives the decimal expansion of 5/13 as 0.38461538461538461538461538461538...

The repeating part is the digits "384615". So, 5/13 = $0.\overline{384615}$.

Example 5. The decimal 0.095 is a terminating decimal, but we *can* write it with an unending decimal expansion if we write zeros for all the decimal places after thousandths:

$$0.095 = 0.095000000000...$$

In other words, we can think of it as repeating the digit zero. In that sense, 0.095 = $0.095\overline{0}$.
However, as you know, we normally write terminating decimals without the extra zeros.

3. Write each decimal using a line over the repeating part.

a. 0.09090909…　　**b.** 5.6843434343…　　**c.** 0.19866666666…

4. Do it the other way around: write the repeating digits several times followed by an ellipsis (three dots).

a. $0.0\overline{887}$　　**b.** $0.245\overline{6}$　　**c.** $2.1\overline{7234}$

5. Which decimal is greater?

a. Which is more, $0.\overline{3}$ or 0.3? How much more?	**b.** Which is more, $0.\overline{55}$ or $0.\overline{5}$? How much more?
c. Which is more, $0.45\overline{0}$ or 0.45? How much more?	**d.** Which is more, $0.\overline{12}$ or 0.12? How much more?

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Two-Step Equations, Part 1

In two-step equations, we need to apply two different operations to both sides of the equation.

Example 1. On the side of the unknown (left), there is a multiplication by 2 and an addition of 3. To isolate the unknown, we need to undo those two operations, in two steps.

$$\begin{aligned} 3x + 2 &= 25 && \mid -2 \\ 3x &= 23 && \mid \div 3 \\ x &= 23/3 \end{aligned}$$

Check:

$$\begin{aligned} 3 \cdot (23/3) + 2 &\stackrel{?}{=} 25 \\ 23 + 2 &\stackrel{?}{=} 25 \\ 25 &= 25 \checkmark \end{aligned}$$

What if you divide first? That is possible:

$$\begin{aligned} 3x + 2 &= 25 && \mid \div 3 \\ \frac{3x+2}{3} &= \frac{25}{3} \\ x + \frac{2}{3} &= \frac{25}{3} && \mid -2/3 \\ x &= 23/3 \end{aligned}$$

Note that this leads to fractions in the middle of the solution process which is more error-prone. Then, the 2 on the left side also has to be divided by 3 (to become 2/3). This is something that is easy to forget and is therefore another reason why subtracting first is the "safer" way, in this case.

If this was a real-life application, we would probably give the answer as a decimal, rounded to a reasonable accuracy. Since it is a mathematical problem, we leave the answer as a fraction. (Why not as a mixed number? It is not wrong, but fractions are less likely to be misread. The mixed number 7 2/3 can easily be misread as 72/3.)

1. Solve. Check your solutions (as always!).

a. $5x + 2 = 67$	**b.** $3y - 2 = 70$	**c.** $3x + 11 = 74$
d. $8z - 2 = 98$	**e.** $75 = 12x + 3$	**f.** $55 = 4z - 11$

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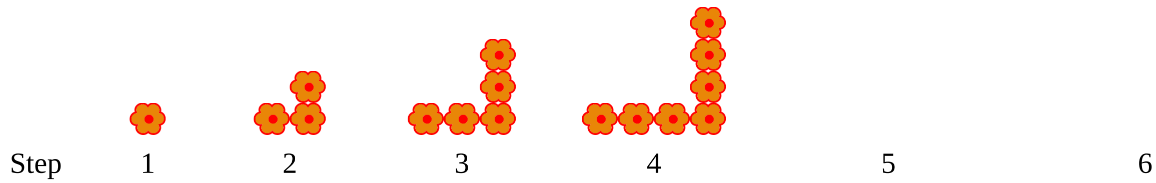
Growing Patterns 2



How do you think this pattern is growing?

How many flowers will there be in step 39?

This pattern adds 2 flowers in each step, except in step 1. This means that by step 39, we have added 2 flowers 38 times. Therefore, there are $1 + 2 \cdot 38 = 77$ flowers in step 39.

Write a formula for the number of flowers in step *n*.

There are several ways to do this. And the three ways explained below are not the only ones!

1. Let's view the pattern as adding 2 flowers in each step after the first one. By step n, the pattern has added one less than n times 2 flowers, because we need to exclude that first step. This means that $(n - 1) \cdot 2$ flowers were added to the one flower that we started with.

   This gives us the expression $1 + (n - 1) \cdot 2$. Since we customarily put the variable first and the constant last, we can rewrite that expression as $1 + 2(n - 1)$ and then as $2(n - 1) + 1$.

2. Another way to think about this pattern is as two legs. One leg includes the flower in the corner, so it has the same number of flowers as the step number. The other leg doesn't have the corner flower, so it has one flower less than the step numbers. In other words, in step 3, we have 3 + 2 flowers. In step 4, we have 4 + 3 flowers. In step 5, we have 5 + 4 flowers.

   This gives us a formula for the number of flowers in step n: there are $n + (n - 1)$ flowers in step n.

3. Yet another way is that, in each step, there are twice as many flowers as the step number, minus one for the flower that is shared. For example, in step 4, we have twice 4 minus 1, which is seven flowers.

   This also gives us a formula: there are $2n - 1$ flowers in step n.

All of the formulas are equivalent (just as we would expect!) and simply represent different ways of thinking about the number of flowers in each step. On the right, you can see how the first two formulas can be simplified to the third one.

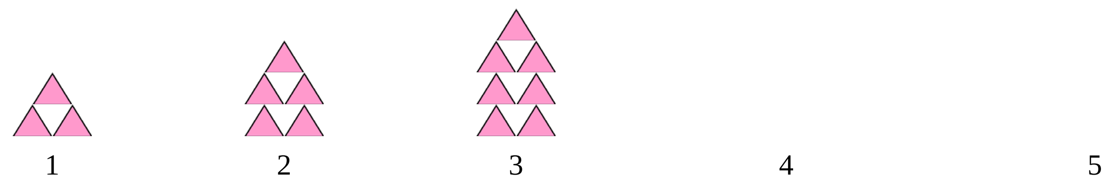
$$\begin{aligned} &n + (n - 1) \\ &= n + n - 1 \\ &= 2n - 1 \end{aligned}$$

$$\begin{aligned} &2(n - 1) + 1 \\ &= 2n - 2 + 1 \\ &= 2n - 1 \end{aligned}$$

In which step are there 583 flowers?

We can use our formula to write an equation to answer this question. In the question, the step number n is unknown, but the total number of flowers in that step is 583. Since we know from our formula that there are $2n - 1$ flowers in step n, we get the equation $2n - 1 = 583$.

$$\begin{aligned} 2n - 1 &= 583 && \mid +1 \\ 2n &= 584 && \mid \div 2 \\ n &= 292 \end{aligned}$$

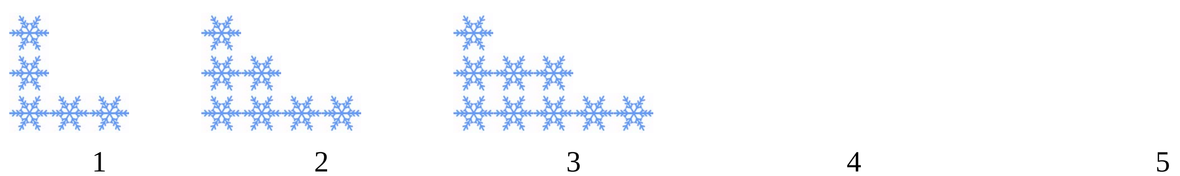


1. **a.** How is this pattern growing?

b. How many triangles will there be in step 39?

c. Write a formula for the number of triangles in step *n*.
Check your answer with your teacher before going on to part (d).

d. In which step will there be 311 triangles?
Write an equation and solve it.
Notice, this question is <u>different</u> from the one in part (c).



2. **a.** How do you think this pattern is growing?

b. How many snowflakes will there be in step 39?

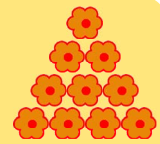
c. Write a formula for the number of snowflakes in step *n*.
Check your answer with your teacher before going on to part (d).

d. In which step will there be 301 snowflakes?
Write an equation and solve it.

Instead of showing the steps of the pattern horizontally, like in the previous exercises, we can also show them like the illustration on the right:

Now, each row of flowers is one step of the pattern.

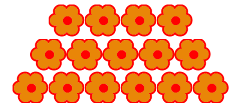
1
2
3
4



. . .

3. A section of a flower garden has rows of flowers as shown on the right.

1
2
3



. . .

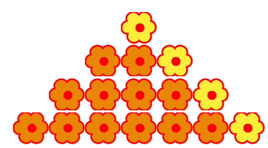
a. Write a formula that tells the gardener the number of flowers in row *n*.

b. How many flowers are in the 28th row?

c. In which row will there be 97 flowers? Write an equation and solve it.

4. Here, we have labeled the first row as "row 0."

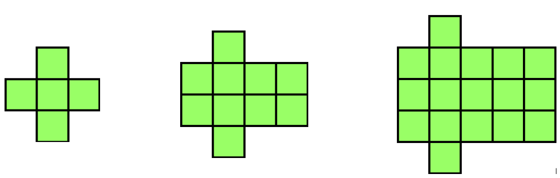
0
1
2
3



. . .

a. Write a formula that tells the gardener the number of flowers in row *n*.

b. In which row will there be 97 flowers? Write an equation and solve it.



Step 1 2 3 4 5

What is the pattern of growth here?
How many squares will there be in step 59?

Puzzle Corner

Using the Distributive Property

Sometimes we need to use the distributive property to remove parentheses.

Depending on the context, this answer might also be given as the decimal 6.2.

Example 1.

$$\begin{aligned} 5(x+9) &= 76 \\ 5x + 45 &= 76 \quad \Big|\ -45 \\ 5x &= 31 \quad \Big|\ \div 5 \\ x &= 31/5 \end{aligned}$$

1. Solve. Give your answers as fractions or whole numbers.

a. $5(x+2) = 85$	**b.** $9(y-2) = 66$	**c.** $2(x-3) = 13$
d. $70 = 8(z+9) - 3z$	**e.** $300 = 20(s-12) + 15s$	**f.** $10(x-9) - 2 = 18$

2. Solve. Now give your answers as decimals, rounded to three decimal digits.

a. $0.4(x+3) = 1.2$	**b.** $30(v-0.4) = -1$	**c.** $0.98 = 3(x-0.07) + 0.9x$

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Contents

Chapter 8: Geometry

Chapter 9: Probability

Chapter 10: Statistics

Foreword

Math Mammoth Grade 7 comprises a complete math curriculum for the seventh grade mathematics studies. It follows the Common Core Mathematics Standards (CCS) for 7th grade. Those standards are so constructed that students can continue to a traditional algebra 1 curriculum after studying this. However, you also have the option of following this course with Math Mammoth Grade 8, which provides a gentler and slower transition to high school math.

The main areas of study in Math Mammoth Grade 7 are:

- The basics of algebra (expressions, linear equations and inequalities);
- The four operations with integers and with rational numbers (including negative fractions and decimals);
- Ratios, proportions, and percent;
- Geometry;
- Probability and statistics.

This book, 7-B, covers ratios and proportions (chapter 6), percent (chapter 7), geometry (chapter 8), probability (chapter 9), and statistics (chapter 10). The rest of the topics are covered in the 7-A worktext.

I heartily recommend that you read the full user guide in the following pages.

I wish you success in teaching math!

Maria Miller, the author

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Ratios and Rates

A **ratio** is a comparison of two numbers, or quantities, using division.

For example, to compare the hearts to the stars in the picture, we say that the ratio of hearts to stars is 5:10 (read "five to ten").

The two numbers in the ratio are called the **first term** and the **second term** of the ratio. The order in which these terms are mentioned does matter! For example, the ratio of stars to hearts is *not* the same as the ratio of hearts to stars. The former is 10:5 and the latter is 5:10.

We can write this ratio in several different ways:

- The ratio of hearts to stars is 5:10.
- The ratio of hearts to stars is 5 to 10.
- The ratio of hearts to stars is $\frac{5}{10}$.
- For every five hearts, there are ten stars.

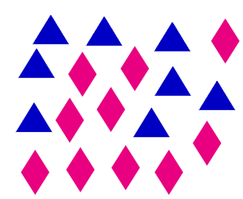
Note that we are not comparing two numbers to determine which one is greater (as in 5 < 10). The comparison is relative as in a multiplication problem. For example, the ratio 5:10 can be simplified to 1:2, and it indicates to us that there are twice as many stars as there are hearts.

We **simplify ratios** in exactly the same way we simplify fractions.

Example 1. In the picture at the right, the ratio of hearts to stars is 12:16. We can simplify that ratio to 6:8 and even further to 3:4. These three ratios (12:16, 6:8, and 3:4) are called **equivalent ratios**.

The ratio that is simplified to lowest terms, 3:4, tells us that for every three hearts, there are four stars.

1. Write the ratio and then simplify it to lowest terms.



The ratio of triangles to diamonds is _____ : _____ = _____ : _____ .

In this picture, there are ______ triangles to every _______ diamonds.

2. **a.** Draw a picture with pentagons and circles so that the ratio of pentagons to the total of all the shapes is 7:9.

b. What is the ratio of circles to pentagons?

3. **a.** Draw a picture in which (1) there are three diamonds for every five triangles, and (2) there is a total of 9 diamonds.

b. Write the ratio of all the diamonds to all the triangles, and simplify this ratio to lowest terms.

4. Write the equivalent ratios.

a. 5 to 45 = 1 to ________	**b.** 3 : _____ = 9 : 60	**c.** 280 : 420 = 2 : _____	**d.** $\frac{5}{13} = \frac{\square}{65}$

We can also form **ratios using quantities that have units**. If the units are the same, they cancel.

Example 2. Simplify the ratio 250 g : 1.5 kg.

First we convert 1.5 kg to grams and then simplify: $\frac{250\text{ g}}{1.5\text{ kg}} = \frac{250\text{ g}}{1{,}500\text{ g}} = \frac{250}{1{,}500} = \frac{1}{6}$.

5. Use a fraction line to write ratios of the given quantities as in the example. Then simplify the ratios.

a. 5 kg and 800 g $\frac{5\text{ kg}}{800\text{ g}} =$	**b.** 600 cm and 2.4 m
c. 1 gallon and 3 quarts	**d.** 3 ft 4 in and 1 ft 4 in

We can generally **convert** ratios with decimals or fractions **into ratios of whole numbers**.

Example 3. Because we can multiply both terms of the ratio by 10, $\frac{1.5\text{ km}}{2\text{ km}} = \frac{15\text{ km}}{20\text{ km}}$.

Then: $\frac{15\text{ km}}{20\text{ km}} = \frac{15}{20} = \frac{3}{4}$. So the ratio 1.5 km : 2 km is equal to 3:4.

You can also see that the ratio is 3:4 by noticing that both 1.5 km and 2 km are evenly divisible by 500 m.

Example 4. Simplify the ratio ¼ mile to 5 miles.

First, the units cancel: ¼ mi : 5 mi = ¼ : 5. Multiplying both terms of the ratio by 4, we get ¼ : 5 = 1:20.

6. Use a fraction line to write ratios of the given quantities. Then simplify the ratios to whole numbers.

a. 5.6 km and 3.2 km	**b.** 0.02 m and 0.5 m
c. 1.25 m and 0.5 m	**d.** 1/2 L and 7 1/2 L
e. 1/4 cup and 3 1/2 cups	**f.** 2/3 mi and 1 mi

If the two terms in a ratio have *different* units, then the ratio is also called a **rate.**
Example 5. The ratio “8 km to 40 minutes” is a rate that compares the quantities “8 km” and “40 minutes,” perhaps for the purpose of giving us the speed at which a person is running.   We can write this rate as 8 km : 40 minutes or $\frac{\text{8 km}}{\text{40 minutes}}$ or 8 km *per* 40 minutes.   The word “per” in a rate signifies the same thing as a colon or a fraction line.
This rate can be simplified: $\frac{\text{8 km}}{\text{40 minutes}} = \frac{\text{1 km}}{\text{5 minutes}}$. The person runs 1 km in 5 minutes.
Example 6. Simplify the rate “15 pencils per 100¢.” Solution: $\frac{\text{15 pencils}}{\text{100¢}} = \frac{\text{3 pencils}}{\text{20¢}}$.

7. Write each rate using a colon, the word “per,” or a fraction line. Then simplify it.

a. Jeff swims at a constant speed of 400 meters : 15 minutes.

b. A car can travel 54 miles on 3 gallons of gasoline.

8. Fill in the missing terms to form equivalent rates.

a. $\frac{\text{1/2 cm}}{\text{30 min}} = \frac{\quad}{\text{1 h}} = \frac{\quad}{\text{15 min}}$	**b.** $\frac{\text{\$88.40}}{\text{8 hr}} = \frac{\quad}{\text{2 hr}} = \frac{\quad}{\text{10 hr}}$

9. Simplify these rates. Don’t forget to write the units.

a. 280 km per 7 hours	**b.** 2.5 inches : 1.5 minutes

10. A car is traveling at a constant speed of 72 km/hour. Fill in the table of equivalent rates: each pair of numbers in the table (distance/time) forms a rate that is equivalent to the rate 72 km/hour.

Distance (km)							
Time (min)	10	30	40	50	60	90	100

11. Eight pairs of socks cost $20. Fill in the table of equivalent rates.

Cost ($)								
Pairs of socks	1	2	4	6	7	8	9	10

Solving Problems Using Equivalent Rates

Example 1. It took Liam 1 ½ hours to paint 8 meters of fence. Painting at the same speed, how long will it take him to paint the rest of the fence, which is 28 meters long?

In this problem, we see a rate of 8 m per 1 ½ hours. There is another rate, too: 28 m per an unknown amount of time. These two are equivalent rates. We can use a table of equivalent rates to solve the problem.

Amount of fence (m)	8	4	28
Time (minutes)	90	45	315

(1) We figure that Liam can paint 4 m of fence in 45 minutes (by dividing the terms in the original rate by 2).

(2) Next we multiply both terms in the rate of 4 m/45 min by seven to get the rate 28 m/315 min.

It will take Liam 315 minutes, or 5 hours 15 minutes, to paint the rest of the fence.

Example 2. Sofia rides her bike 20 miles in 45 minutes. Riding at the same speed, how far will she go in 1 hour?

$$\frac{20\text{ miles}}{45\text{ minutes}} \overset{\div 3}{=} \frac{\boxed{}\text{ miles}}{15\text{ minutes}} \overset{\times 4}{=} \frac{\boxed{}\text{ miles}}{60\text{ minutes}}$$

We can multiply or divide both terms of a rate by the same number to form another, equivalent rate. (You have used this same idea in the past with equivalent fractions.)

It's not easy to go directly from 45 minutes to 60, but we can use 15 as a "stepping stone" in between.

Recall that 20 ÷ 3 is easy to solve when you think of it as a fraction: 20/3 = 6 2/3. Sofia can ride 6 2/3 miles in 15 minutes.

Then, we multiply both terms of that rate by 4. Again, don't be intimidated by the fraction: 4 · (6 2/3) = 4 · (20/3) = 80/3 = 26 2/3. So, Sofia can ride 26 2/3 miles in 1 hour.

1. Fill in the tables of equivalent rates.

a.

Distance	15 km			
Time	3 hr	1 hr	15 min	45 min

b.

Pay	$15			
Time	45 min	15 min	1 hr	1 hr 45 min

2. Fill in the missing terms in these equivalent rates.

a. $\frac{3\text{ pies}}{8\text{ boys}} = \frac{\_\_\_\_\_}{2\text{ boys}} = \frac{\_\_\_\_\_}{12\text{ boys}} = \frac{\_\_\_\_\_}{20\text{ boys}}$

b. $\frac{115\text{ words}}{2\text{ min}} = \frac{\_\_\_\_\_}{1\text{ min}} = \frac{\_\_\_\_\_}{3\text{ min}}$

3. Aiden can ride his bicycle 8 miles in 28 minutes. At the same constant speed, how long will he take to go 36 miles?

$$\frac{8\text{ miles}}{28\text{ minutes}} = \frac{4\text{ miles}}{\boxed{}\text{ minutes}} = \frac{\boxed{}\text{ miles}}{\boxed{}\text{ minutes}}$$

Example 3. You get 20 erasers for $3.80. How much would 22 erasers cost (at the same rate)?

One way to solve this is to calculate the cost for ONE eraser, or **the unit rate**, and then multiply that by 22:

$$\frac{\$3.80}{20 \text{ erasers}} \overset{\div 20}{=} \frac{\$0.19}{1 \text{ eraser}} \overset{\times 22}{=} \frac{\$4.18}{22 \text{ erasers}}$$

Another way is to find the cost for two erasers first.

Price	$3.80	$0.38	$4.18
Erasers	20	2	22

Twenty erasers cost $3.80, so 2 erasers cost 1/10 that much, or $0.38. Lastly, the cost of 22 erasers is the cost of 20 + 2 erasers, or $3.80 + $0.38 = $4.18.

4. If 15 muffins cost $10.80, how much would 40 muffins cost?

5. See Lucas's and Avery's solutions to a problem about pencils below. One of them must be in error. Check their work and find the error.

A set of 30 pencils costs $4.50. Is that equal to the rate of 50 pencils for $7.25? If not, which pencils are cheaper?

Lucas: I will calculate the unit rates.

1st set: $4.50/30 = $0.15 per pencil

2nd set: $7.25/50 = $0.155 per pencil

The pencils in the first set are cheaper per pencil.

Avery: I will calculate the price for 150 pencils using both rates.

1st set:

Price	$4.50	$22.50
Pencils	30	150

2nd set:

Price	$7.25	$21.75
Pencils	50	150

The pencils in the second set are cheaper.

6. **a.** A train travels at a constant speed of 111 km per hour. How far will it go in 140 minutes?

b. Is this equal to the rate of traveling 90 km in 40 minutes?

7. Mason earns $220 for eight hours of work. In how many (whole) hours will he earn at least $600?

Example 4. For each hour of computer work, you are supposed to do simple exercises for 5 minutes.

Since 1 hour is 60 minutes, we can write this rate as 5 minutes : 60 minutes.

Notice how the rate has the *same* units in both terms. Those units can therefore be canceled, and the rate (really, a ratio) is equivalent to the unitless ratio 5 : 60 = 1 : 12. This means you are supposed to spend 1/12 of the time doing exercises when doing computer work.

8. Refer to Example 4. How much time should you spend doing exercises for 4 1/3 hours of computer work?

9. A certain medicine for dogs is administered at the rate of 5 mg per each kilogram of body weight.

a. You have 125 mg of that medicine. What is the maximum weight for a dog you could treat with it?

b. Another medicine for the same problem is administered at the rate of 20 mg per each 5 kg of body weight. If your dog weighs 13 kg, which of the two medicines would you need more of?

10. A car uses 3.9 liters of gasoline to travel 45 km. How many liters of gasoline would the car need for a trip of 60 km?

11. In a poll that interviewed 1,000 people about their favorite color, 640 people said they liked blue.

a. Simplify this ratio to lowest terms.

b. Assuming the same ratio holds true in another group of 125 people, how many of those people can we expect to like blue?

12. Jane and Stacy ran for 30 seconds. Afterward each girl checked her heartbeat. Jane counted that her heart beat 38 times in 15 seconds, and Stacy counted that her heart beat 52 times in 20 seconds. Which girl had a faster heart rate, measured in bpm (beats per minute)? How much faster?

Unit Rates

Remember that a rate is a ratio where the two terms have different units, such as 2 kg/$0.45 and 600 km/5 hr.

In a **unit rate, the second term of the rate is one** (of some unit).

For example, 55 mi/1 hr and $4.95/1 lb are unit rates. The number “1” is nearly always omitted so those rates are usually written as 55 mi/hr and $4.95/lb.

To convert a rate into an equivalent unit rate simply divide the numbers in the rate.
This means you divide the first term of the rate by the second term of the rate.

Example 1. Mark rides his bike 35 km in 1 ½ hours. What is the unit rate?

We are given the rate 35 km : 1 ½ hr. To find the unit rate, we *divide* 35 km by 1 ½ hr.

Note how the units “km” and “hours” are divided, too, and become “km per hour” or “km/hour.”

$$\frac{35\text{ km}}{1\ 1/2\text{ h}} = 35 \div \frac{3}{2}\text{ km/hr} = 35 \cdot \frac{2}{3}\text{ km/hr}$$

$$= \frac{70}{3}\text{ km/hr} = 23\tfrac{1}{3}\text{ km/hr.}$$

We could also use decimal division:
35 km ÷ 1.5 hr = 23.333... km/hr.

So, the unit rate is 23 ⅓ km per hour.

Example 2. A snail can slide through the mud 5 cm in 20 minutes. What is the unit rate?

We simply divide 5 cm ÷ 20 min. Notice how the units cm and min also get divided:

$$\frac{5\text{ cm}}{20\text{ min}} = \frac{5}{20}\text{ cm/min} = \frac{1}{4}\text{ cm/min}$$

As a decimal, this is 0.25 cm/min.

Here, the unit rate is the snail’s speed, and considering we got such a small number, it is more practical to express it in centimeters per *hour*.

To do that, we can multiply the 1/4 cm/min by 60. Or, we can multiply both terms of the *original* rate by 3 to get the equivalent rate of 15 cm : 60 min, or 15 cm per hour. He’s not going very fast!

1. Find the unit rate.

a. $125 for 5 packages

b. $6 for 30 envelopes

c. $1.37 for ½ hour

d. 2 ½ inches per 4 minutes

e. 24 m^2 per 3/4 gallon

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Proportional Relationship or Not?

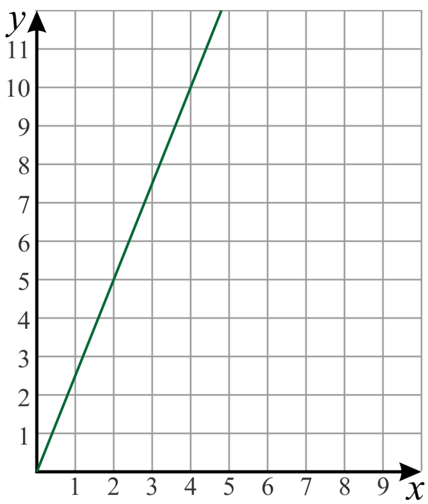
You have learned that if two variables are in a proportional relationship, then the rates formed by the values of the variables are equivalent. This means that if the value of one doubles, the value of the other doubles also. If one quantity decreases to 1/5 of its value, the other does the same. In fact, if one value is multiplied or divided by any number, the same happens to the other (with the exception of the point (0, 0)).

Now we will learn something about the graph of an equation that depicts a proportional relationship.

1. Find the two situations below that show the variables to be in a proportional relationship.
   How do the graphs of those equations differ from the graphs depicting a *non*-proportional relationship?

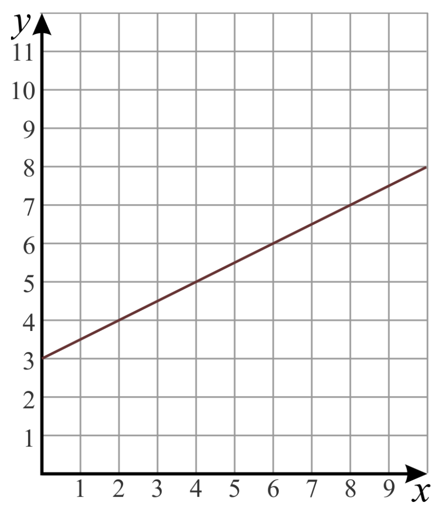
(i)
$y = 2.5x$

y	0	2.5	5	7.5	10	12.5
x	0	1	2	3	4	5



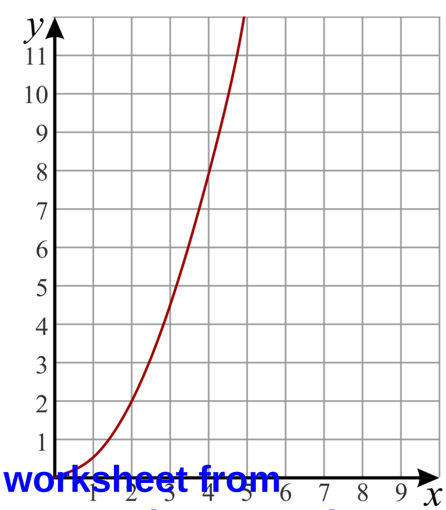
(ii)
$y = 0.5x + 3$

y	3	3.5	4	4.5	5	5.5
x	0	1	2	3	4	5



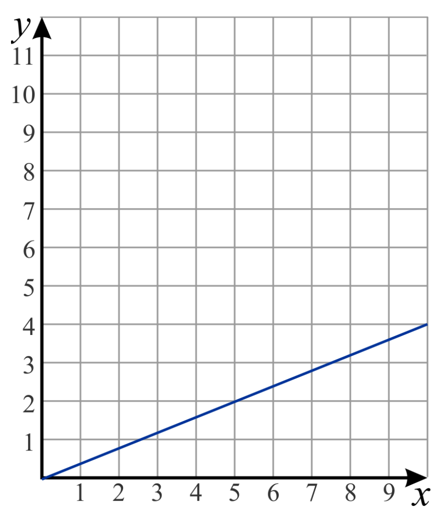
(iii)
$y = 0.5x^2$

y	0	0.5	2	4.5	8	12.5
x	0	1	2	3	4	5



(iv)
$y = 0.4x$

y	0	0.4	0.8	1.2	1.6	2
x	0	1	2	3	4	5



You can check to see if two variables are in direct variation in several different ways.

(1) Check if the <u>values</u> of the variables are in direct variation (if the rates are equivalent).
If you double the value of one, does the value of the other double also? Or, maybe one rate is $42/6 kg and the other is $35/5 kg. Those are equivalent rates, both being equal to the unit rate of $7/kg.

(2) When two variables are proportional, the equation relating the two is of the form $y = mx$, where y and x are the variables, and m is a constant. If the equation is of some other form, such as $y = 2x^2$ or $y = 1/x$ or $y = mx + b$, the variables are not proportional.

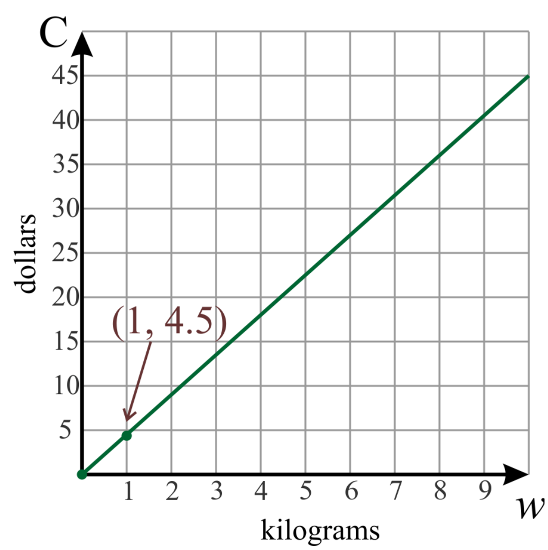
(3) The graph of such an equation is always a line through the origin.

Example. Apples cost $4.50 per kg. The equation relating the cost (C) and the weight (w) of apples is $C = 4.5w$, which is of the form $y = mx$. The quantities are proportional.

Note the point (1, 4.5) that corresponds to the unit rate. The other special point on the graph is the origin at (0, 0). That point is always on the graph of a proportional relationship.

It is possible to look at the situation just the opposite way, and consider how the weight of the apples depends on the cost of the apples. In that case, we would write the equation $w = \frac{1}{4.5}C$, which is approximately $w = 0.22C$, and plot the values of C on the horizontal axis. This way is not as common as observing how the cost depends on the weight.

Cost of Apples



You might wonder, "Why does the line have to go through the origin if the quantities are in proportion?"

Consider this principle governing direct variation: if one quantity is cut in half, then the other is cut in half. Let's say you start with certain values of the two quantities, such as 6 meters per 2 minutes. Now cut both in half to get 3 meters per 1 minute. Do it again to get 1.5 meters per 1/2 minute, then 0.75 m per 1/4 minute. Do it again, and again.

Notice that both numbers get smaller and smaller yet—they approach zero. This would happen no matter what values of the two quantities you started with. So the point (0, 0) has to be included in the graph of quantities that are in direct variation.

2. Determine whether the two variables are proportional. If so, write an equation relating them.

a.

b	2	3	4	5	6
a	0	1	2	3	4

b.

y	0	4	8	12	16
x	0	1	2	3	4

c.

t	0	1/3	2/3	1	4/3
s	0	1	2	3	4

d.

w	20	15	10	5	0
v	0	1	2	3	4

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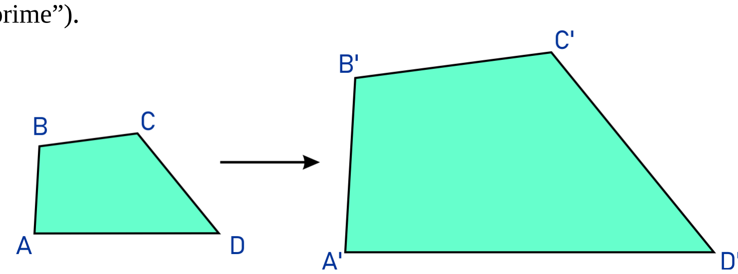
Scaling Figures

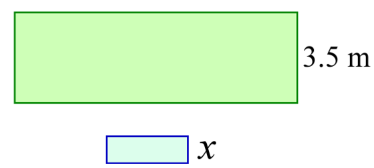
Example 1. Trapezoid ABCD has been enlarged or **scaled** proportionally to become trapezoid A'B'C'D'. (Note: A' is read as "A prime").



The two trapezoids have the same basic shape, but one is bigger than the other.
We say such figures are **similar figures**, and that the one is a **scaled image** of the other.

When we **scale a figure**, we enlarge or shrink it while maintaining its shape. In this process, all the dimensions of the figure are multiplied by the same number called the **scale factor** or just the **scale**.

Example 2. The bigger rectangle was shrunk using a scale factor of 2/7. Find the length of the side marked *x*.

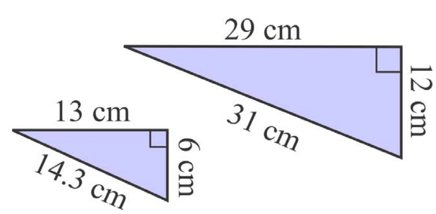


We simply multiply the given length 3.5 m by the scale factor:

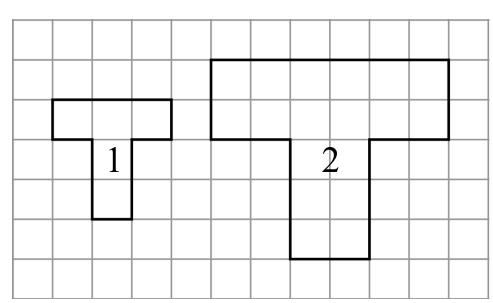
(2/7) · 3.5 m = 2 · 3.5 m / 7 = <u> 1 m </u>

1. Find the scale factor used in Example 1 to enlarge trapezoid ABCD. Do so by measuring the sides of the trapezoids with a millimeter-ruler.

2. Are the two triangles similar? Why or why not?
(Note: the images are not to scale.)

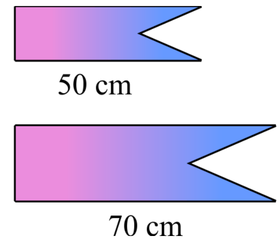


3. Is Figure 2 a scaled image of figure 1? Explain.



Scale factor and similarity ratio

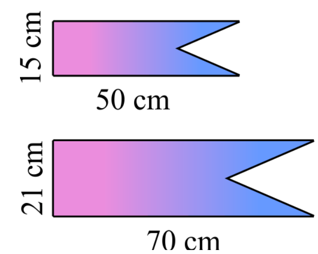
Example 3. The smaller figure was enlarged. What is the **scale factor**?



Since the 50 cm-side became 70 cm long, the scale factor from the smaller figure to the larger one is 70/50 = 7/5 = 1.4.
So, each side of the figure became 1.4 times as long as before.

(Note that the scale factor is *not* 5/7, because that is less than 1, and would signify that the figure shrunk.)

We can also consider the **similarity ratio** (or ratio of magnification): it is the ratio of any side of the figure and the corresponding side in the scaled figure.



Using the 50-cm and 70-cm sides, we get that the similarity ratio of the smaller shape to the bigger shape is 50:70 = **5:7**.

If we use the other marked sides, we get the same: 15:21 = **5:7**.
This ratio is the same, no matter which side or dimension we use.

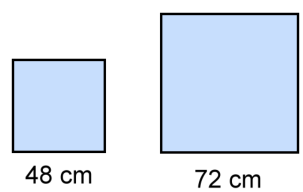
If we want the similarity ratio of the bigger shape to the smaller shape, we list the bigger shape's side length first. The similarity ratio is then 70:50 = 7:5.

So, the scale factor was 7/5 or 1.4, and the similarity ratio was 5:7 or 7:5, depending on whether you list it as of the smaller shape to the bigger shape or vice versa.

Note that the scale *factor* is a single **number** (such as 3), but the similarity *ratio* is a **ratio** of two numbers (such as 3:1). Always be careful to note which way you use the scale factor or the similarity ratio. If the figure gets bigger, then multiply by a number that is more than 1. If it gets smaller, multiply by a number that is less than 1.

4. The smaller square was enlarged to become the larger square.
   What is the scale factor?

   The similarity ratio?



5. A rectangle with 20 ft and 48 ft sides is shrunk proportionally so that its shorter side becomes 15 ft.

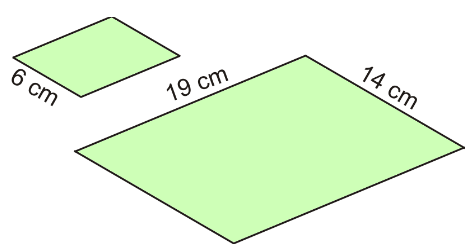
   a. What is the similarity ratio?

   b. What is the scale factor?

   c. How long is the longer side in the smaller rectangle?

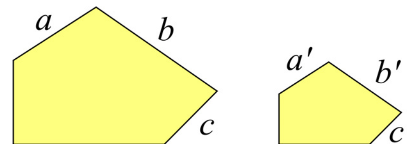
6. **a.** Find the scale *factor* when the smaller parallelogram is enlarged to become the bigger one.

   b. What is the similarity *ratio*?



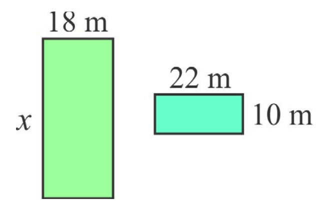
When two figures are similar, the **corresponding dimensions of the original image and the scaled image are proportional**. In other words, they are in the same ratio — and this ratio is the similarity ratio.

The illustration on the right shows two similar pentagons, with some side lengths marked. The ratio *a* : *a′* equals the ratio *b* : *b′*, and both of them equal the ratio *c* : *c’* — and the same would be true for the other corresponding sides.



As an equation, $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

Example 4. Sometimes you need to look carefully to find the corresponding sides. The two rectangles at the right are similar. Notice that the “top” sides of 18 m and 22 m do *not* correspond. Instead, the 18 m side corresponds to the 10 m side, because they are the *shorter* sides of the rectangles.



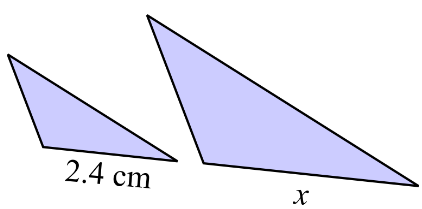
The ratio of 18 m : 10 m = 9:5 is the similarity ratio. It is equal to the ratio *x* : 22 m we can write using the other corresponding sides.

This means we can use a proportion to solve for *x* (on the right).

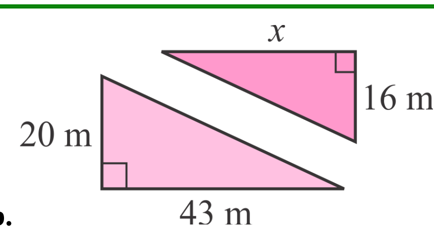
$$\frac{x}{22\text{ m}} = \frac{9}{5}$$
$$198\text{ m} = 5x$$
$$x = 39.6\text{ m}$$

7. The figures are similar. Find the length of the side labeled *x*.

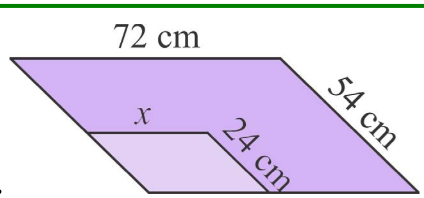
a. Similarity ratio 3:5.



b.



c.



8. The sides of two similar triangles are in a ratio of 3:4. If the sides of the larger triangle are 4.8 cm, 6.0 cm, and 3.6 cm, what are the sides of the smaller triangle?
Hint: Here you don't need a proportion since the numbers are easy. You can reason logically.

9. The sides of a rectangle measure 12 cm and 18 cm. The shorter side of another, similar rectangle is 4 cm.

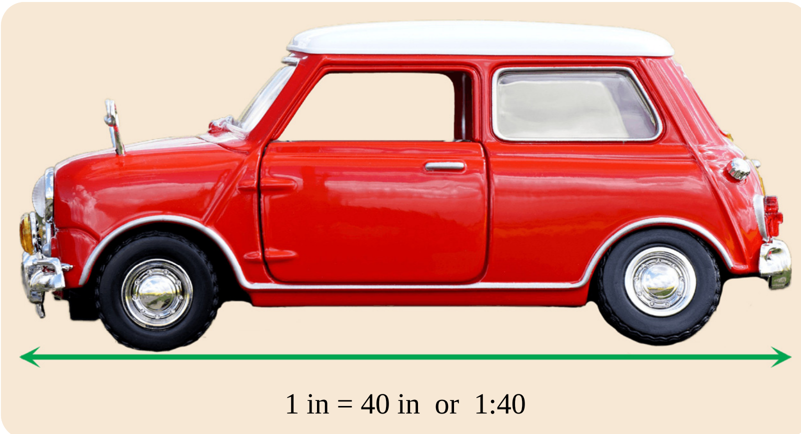
a. In what ratio are the sides of the two rectangles?

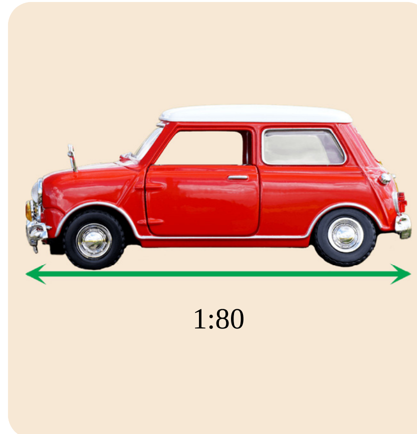
b. Calculate the areas of both rectangles.

c. (optional) In what ratio are their areas?

10. (optional) Draw any triangle on blank paper or below. Then draw another, bigger triangle using the scale ratio of 2:5. You will probably need to measure some angles from your triangle to be able to draw it. Note that **corresponding *angles* in the two triangles will be equal**. Only the side lengths change — not the angles.

Scale Drawings 1





Example 1. Here you see two scaled images of a car, with two different scales. The left image has the scale of **1:40**, which means that 1 inch in the picture corresponds to 40 inches in reality.

In fact, the scale is also written as "**1 in = 40 in**". This doesn't mean that 1 inch really is as long as 40 inches; that wouldn't make sense. It means that 1 inch *in the image* equals 40 inches *in reality*.

The image on the right has the scale of 1:80. One unit in the image corresponds to 80 units in reality.

Notice how, for the picture on the right, the number in the scale got *bigger*, but the picture got *smaller*. How much smaller did it get?

Find how long the car is in reality. Use a ruler.

Do you get the same answer using either picture and its scale?

You can use a calculator for all the problems in the lesson.

1. **a.** Let's say you are to draw a third picture of the car in Example 1, with a scale of 1:160. How can you use the scale of 1:80 and the new scale of 1:160 to figure out the dimensions (width and length) of the new drawing, without figuring it from the dimensions in reality?

   b. How long will the car be if drawn to the scale of 1:160? How tall?

2. This rectangle shows a plot of land, drawn here at the scale of 1:2000.

   a. Measure its dimensions, and then redraw it at the scale of 1:500.
   (Will it be bigger or smaller than the image above?)

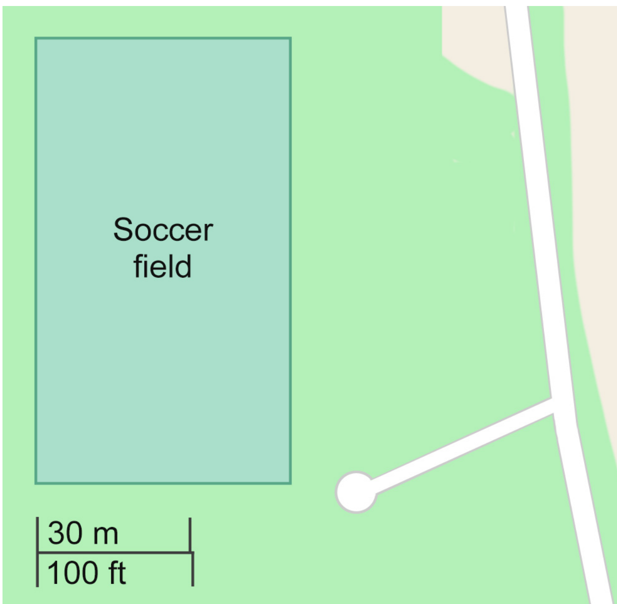
   b. Find its dimensions and its area in reality.

3. **a.** Look at the scale on the map of the soccer field below. The line indicating 30 m on the map is 2 cm long.*
Rewrite this scale of **2 cm = 30 m** as a unitless ratio, in the form 1: (some number).
(Hint: First make sure both measurements are in the same unit.)
*It is 2 cm long if the page was printed at 100%.

b. If the soccer field was redrawn with the scale 1 cm = 30 m, would it be bigger or smaller than the picture here?

By how much?

c. Draw another scale drawing of the soccer field on the right, using the scale 1 cm = 20 m.



Draw the soccer field here, using the scale 1 cm = 20 m.

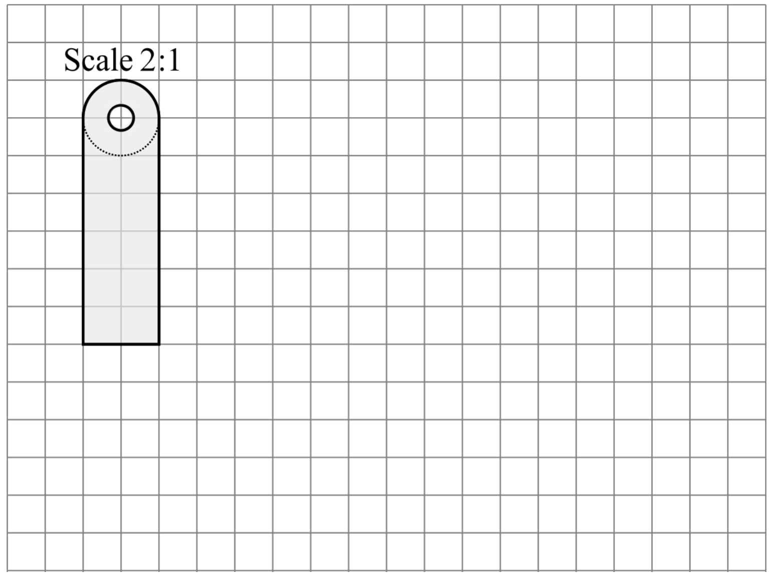
d. Find the length, width, perimeter, and area of the soccer field in reality. Use meters and square meters.

4. Your family is purchasing an air conditioner. In the store, you take photos of the AC units on a phone. One unit is 32 inches wide by 12 inches tall in reality. On the phone, in the picture, the unit is 6 in by 2 1/4 in. What is the scale? What is the scale *factor* (from the picture to reality)?

5. The picture shows a metal part designed for a machine.

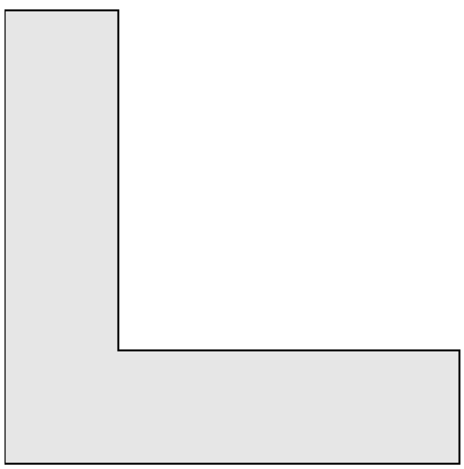
   a. Redraw the part in the scale 1:1.

   b. Redraw the part in the scale 3:1.



6. The L-shape depicts a driveway to Shaun's house. It is drawn here at the scale of 1:300.

   a. Draw another scale drawing of the driveway at the scale of 1:400.



   b. Measure the necessary dimensions of the driveway, and find its actual area in square meters.

   c. Gravel is sold by cubic yard, and it will be spread one foot deep, so the calculation for the volume of the driveway needs to be done in cubic feet, then converted to cubic yards.

   (i) Convert your answer from (b) to square feet using this factor: 1 m^2 = 10.7639 square feet.

   (ii) The gravel will be applied 1 foot deep. Now find the volume of the needed gravel in cubic feet.

   (iii) Convert this to cubic yards, using 1 cubic yard = 27 cubic feet.
   Round your answer *up* to the nearest half cubic yard.

   (iv) Gravel costs $65 per cubic yard. Now calculate the cost of the gravel for the driveway.

Floor Plans

Floor plans are drawn using a **scale**, which is a ratio relating the distances in the plan to the distances in reality. For example, a scale of 1 cm : 2 m means that 1 cm in the drawing corresponds to 2 m in reality.

Example 1. A room measures 1 ¾″ by 2 ½″ in a plan with a scale of 1 in: 10 ft. How big is it in reality?

Since 1 inch corresponds to 10 ft, we simply need to multiply the length and width given in inches by 10 to get the dimensions in feet.

Using decimals, the dimensions are 1.75 in by 2.5 in. So, the dimensions of the room in actual size are 1.75 · 10 = 17.5 ft and 2.5 · 10 = 25 ft.

However, we're really not just multiplying by the number 10 but by the ratio 10 ft/1 in. That is how we keep track of the units to make sure that our final answer ends up with the correct units (feet and not inches). This is what's really happening in the calculation:

$$1.75\ \cancel{\text{in}} \cdot \frac{10\text{ ft}}{1\ \cancel{\text{in}}} = 17.5\text{ ft} \quad \text{and} \quad 2.5\ \cancel{\text{in}} \cdot \frac{10\text{ ft}}{1\ \cancel{\text{in}}} = 25\text{ ft}$$

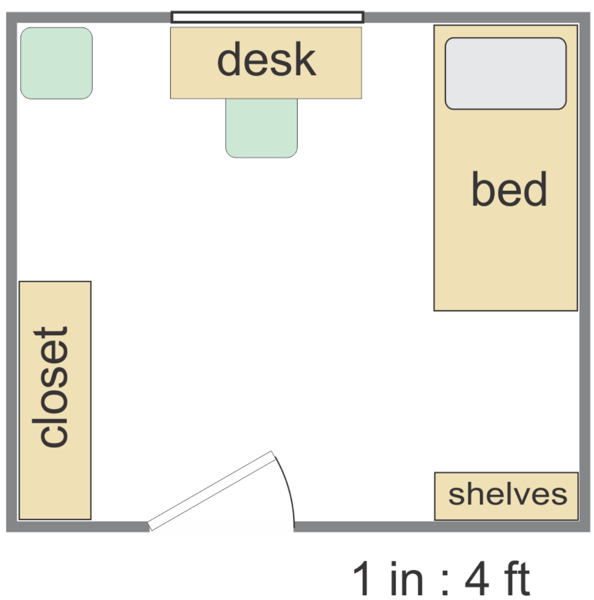
Why not multiply by 1 in/10 ft (or 1 in : 10 ft) as the ratio is stated in the problem? Then the inches ("in") in the dimension wouldn't cancel the inches in the conversion factor, and we would end up "in^2/ft" as our unit of length, instead of "ft."

1. This room is drawn at a scale of 1 in : 4 ft. This means that 1 in on the plan corresponds to 4 ft in reality. It also means you can simply multiply or divide by 4 to convert dimensions between the plan and reality and vice versa.

   Measure the dimensions for the bed and the desk from the picture. Then calculate the actual (real) dimensions.

   a. the bed

   b. the desk



2. What is the area of this room in reality?

3. In the middle of the plan for the room, draw a table that in reality measures 3.5 ft × 2.5 ft.

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Review: Percent

Percent (or **per cent**) means *per hundred* or "divided by a hundred." (The word "cent" means one hundred.) So, percent means the rate per hundred, or a hundredth part.

To convert percentages into fractions, simply read the "per cent" as "per 100." Thinking of hundredths, you can also easily write them as decimals.

Therefore, 8% = 8 per cent = 8 per 100 = 8/100 = 0.08.

Similarly, 167% = 167 per 100 = 167/100 = 1.67.

$\frac{5}{100}$ five per cent = 5%

1. Write as percentages, fractions, and decimals.

Percent	Decimal	Fraction
	0.07	
52%		
		$\frac{59}{100}$

Percent	Decimal	Fraction
109%		
200%		
		$\frac{382}{100}$

A number with two decimal digits has hundredths, so it can easily be written as a percentage. For example, 0.56 = 56%. But we can write numbers with more decimal digits as percents, also.

Example 1. As a percentage, the number 0.5642 is 56.42%. Compare this to 0.56 = 56%. The digits "42" simply follow the digits "56", and become the decimal digits for the percentage.

Decimal	Percent	Fraction
0.09	9%	$\frac{9}{100}$
0.091	9.1%	$\frac{91}{1000}$
0.09146	9.146%	$\frac{9146}{100{,}000}$

2. Write as percentages, fractions, and decimals.

Percent	Decimal	Fraction
0.9%		
		$\frac{282}{1000}$
	0.8914	

Percent	Decimal	Fraction
		$\frac{91}{10{,}000}$
2.391%		
	0.94284	

Writing fractions as percentages

Example 2. Sometimes you can convert a fraction into an equivalent fraction with a denominator of 100, 1000, or some other power of 10. After that it is easy to write it as a decimal and then as a percentage.

$$\frac{46}{25} \underset{\cdot 4}{\overset{\cdot 4}{=}} \frac{184}{100} = 1.84 = 184\%$$

Example 3. For most fractions, we need to use *division* to convert the fraction to a decimal first, and then to a percentage.

Simply treat the fraction line as a division symbol and divide (using long division or a calculator), to get a decimal. Then write it as a percentage.

$$\frac{8}{9} = 0.888... \approx 0.889 = 88.9\%$$

```
   0.8888
9 )8.0000
  -7 2
     8 0
   - 7 2
       8 0
     - 7 2
         8 0
       - 7 2
           8
```

3. Fill in the table. First write each fraction as an equivalent fraction where the denominator is a power of ten.

Fraction	Fraction (denominator is a power of ten)	Decimal	Percent
$\frac{8}{25}$	$\frac{\square}{100}$		
$\frac{142}{200}$	$\frac{\square}{100}$		
$\frac{24}{20}$			
$\frac{31}{250}$			
$\frac{3}{8}$			

4. Write as percentages. Use long division. Round your answers to the nearest tenth of a percent.

a. 11/8	**b.** 11/24

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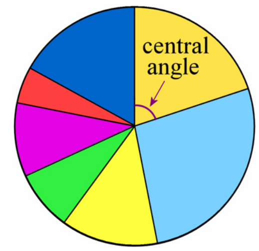
Circle Graphs

(This lesson is optional.)

A **circle graph** shows visually how a total is divided into parts (percentages). Each of the parts (pie slices) is a **sector**, and each sector has a **central angle**.

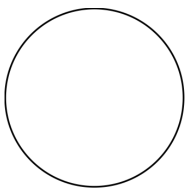


To make a circle graph, we need to calculate the measure of the central angle for each sector. For example, if a circle graph is supposed to show the percentages 25%, 13%, and 62%, we calculate those percentages of 360º (the full circle):

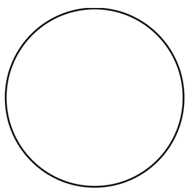
25% of the total corresponds to 0.25 · 360º = 90º.
13% of the total corresponds to 0.13 · 360º = 46.8º.
62% of the total corresponds to 0.62 · 360º = 223.2º.

1. Sketch a circle graph that shows...

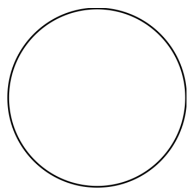
a. 50%, 25%, and 25%



b. 33.3%, 33.3%, 1/6, and 1/6

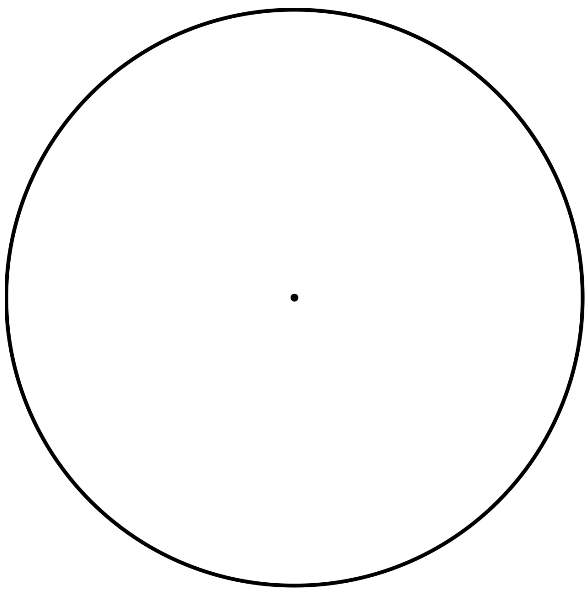


c. 20%, 20%, 10%, and 50%

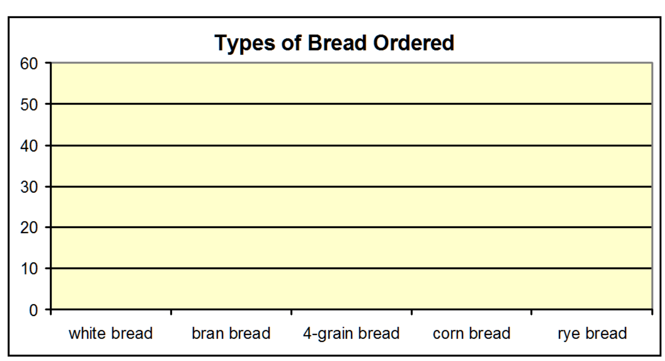


2. The table shows different kinds of specialty breads that a grocery store ordered. Fill in the table. Make a circle graph.
(*Note:* You will need a protractor to draw the angles.)

Type	Quantity	Percentage	Central Angle
white bread	50		
bran bread	25		
rye bread	30		
corn bread	40		
4-grain bread	55		
TOTALS	**200**	**100%**	**360º**

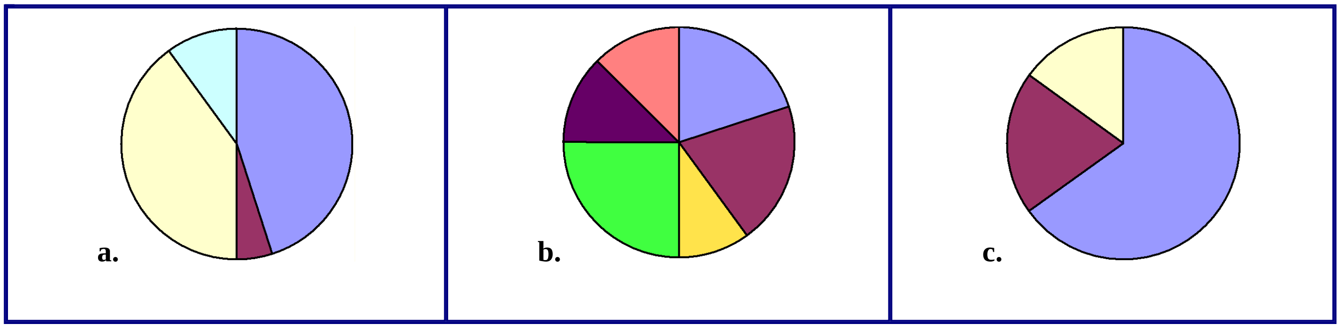


3. Make a bar graph of the quantities of each type of bread from the table above. →



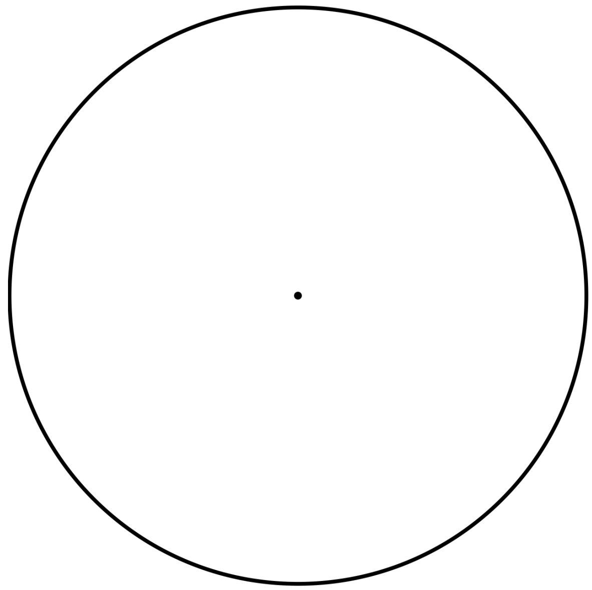
Does the bar graph show percentages?

4. Think of fractions. Estimate the percentages that the sectors of the circle represent.



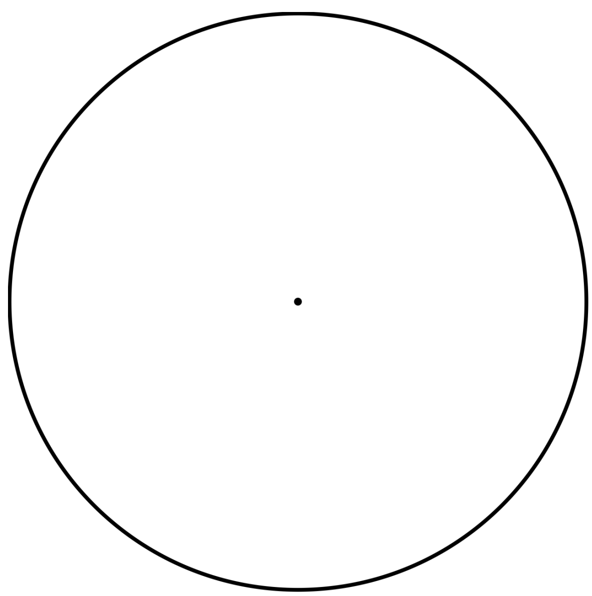
5. The table lists by flavor how many units of protein powder a company sold. Draw a circle graph showing the percentages. You will need a protractor and a calculator.

Flavor	Amount sold	Percentage of total	Central Angle
chocolate	67		
vanilla	34		
strawberry	16		
blueberry	26		
TOTALS		**100%**	**360º**



6. Mark polled some seventh graders about their favorite hobbies. Draw a circle graph to show the percentages. Round the angles to whole degrees. You will need a protractor.

Favorite hobby	Percentage	Central Angle
Reading	12.3%	
Watching TV	24.5%	
Computer games	21%	
Sports	22.3%	
Pets	7.1%	
Collecting	8.1%	
no hobby	4.7%	
TOTALS	**100%**	**360º**



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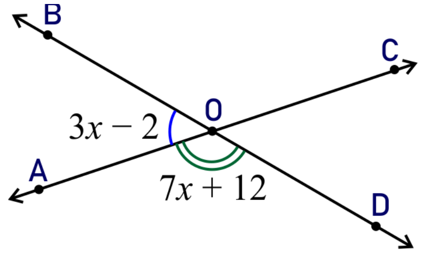
Angle Relationships 3

Example 1. Two lines intersect at O. Find the measure of ∠AOD.



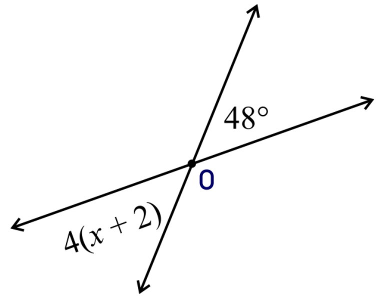
We can write an equation to solve for x, based on the fact that angles AOB and DOA are supplementary, thus sum to 180°:

$$\begin{aligned}(3x-2)+(7x+12) &= 180\\ 3x-2+7x+12 &= 180\\ 10x+10 &= 180\\ 10x &= 170\\ x &= 17\end{aligned}$$

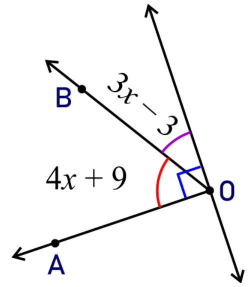
Now, 17 is not yet the final answer, because ∠AOD does not measure x°. It measures $7x + 12$ degrees!

We calculate: $7x + 12 = 7(17) + 12 = 131$. So, ∠AOD = 131°.

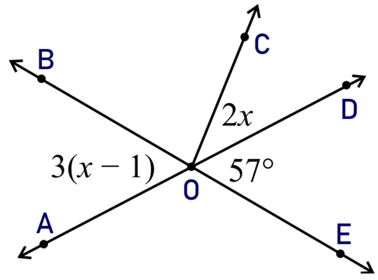
1. Two lines intersect at O. Find the value of x.



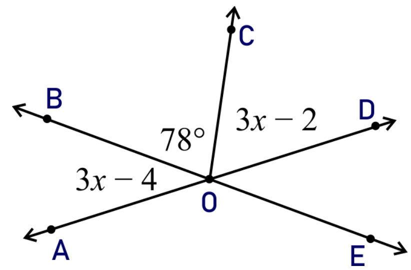
2. Find the measure of ∠AOB.



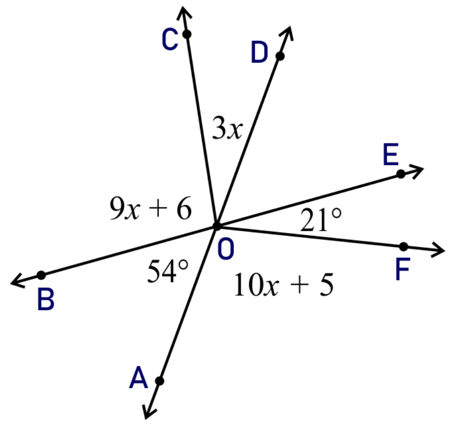
3. Lines AD and BE intersect at O. Find the measure of ∠COD.



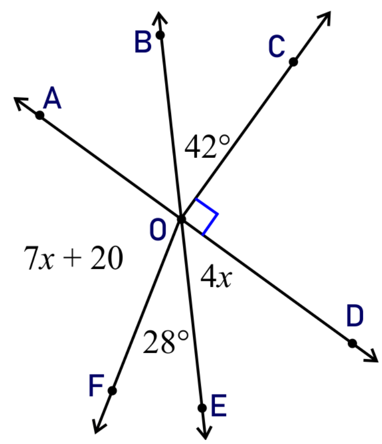
4. Lines AD and BE intersect at O. Find the measures of ∠AOE and ∠DOE.



5. Lines AD and BE intersect at point O which is also the starting point for rays OC and OF.
Find the measures of ∠BOC and ∠DOE.



6. Lines AD and BE intersect at point O which is also the starting point for rays OC and OF.
Find the measures of ∠AOB and ∠AOF.



Puzzle Corner

In an isosceles right triangle, the top angle measures $2x + 5$ degrees. Find the value of x.

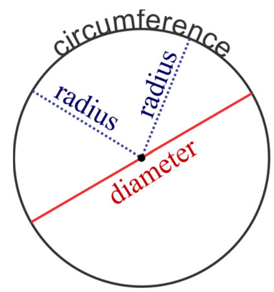
Circumference of a Circle

Circle terms

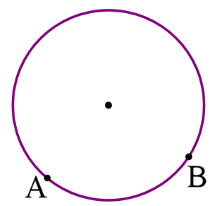
- The **circumference** of a circle is the perimeter, or outside curve, of the circle.
- The **radius** is any line segment from the center point to the circumference.

In fact, what makes a circle, a circle is the fact that all the points on the circumference are **at the same distance from the center point** of the circle. This distance is called the **radius** of the circle.

- The **diameter** is any line segment from circumference to circumference that goes through the center point of the circle.



You may use a calculator for every problem in this lesson.



1. **a.** Draw a radius for this circle from the point A and a diameter from the point B.

   b. What simple relationship exists between the diameter and the radius of any circle?

2. A circle is drawn on the ground. Your 15-foot jump rope is just enough to go around it. Match the approximate measurements and the terms.

diameter	about 15 ft
radius	about 2.5 ft
circumference	about 5 ft

3. There exists an amazing relationship between the circumference and the diameter of every circle! Let's study it now. Find at least **five circular objects**, such as a plate, a can, a glass, and so on. Measure the diameter (*d*) of each circle with a ruler. Measure the circumference (C) of each circle by placing a string around the object, and then measuring the length of the string. Record your results in the table.

   In the last column, divide the circumference by the diameter using a calculator (separately for each object). In other words, you will calculate the ***ratio* of the circumference to the diameter**.

Object	C	*d*	C ÷ *d*

What do you notice?

If you have measured accurately, for each object, the ratio of C to *d* should be a little over 3.

Probability

You *probably* already have an intuitive idea of what *probability* is. In this lesson we look at some simple examples in order to study probability from a mathematical point of view.

If we flip a coin, the chance, or **probability**, of getting “heads” is 1/2. The chance of getting “tails” is also 1/2. “Heads” and “tails” are the two possible **outcomes** when tossing a coin, and they are equally likely.

When rolling a six-sided number cube (a die), you have six possible **outcomes**: you can roll either 1, 2, 3, 4, 5, or 6. These are all equally likely (assuming the die is fair).

Thus the probability of rolling a five is 1/6. The probability of rolling a three is also 1/6. In fact, the probability of each of the six outcomes is 1/6.

The probability of rolling an even number is 3/6, or 1/2, because three of the six possible outcomes are even numbers.

Simple probability has to do with situations where each possible outcome is <u>equally likely</u>.

Then the **probability** of an event is the fraction $\dfrac{\textbf{number of favorable outcomes}}{\textbf{number of possible outcomes}}$

“Favorable outcomes” are those that make up the event you want. The examples will make this clear.

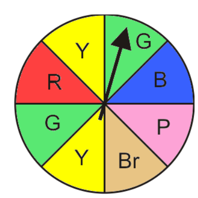
Example 1. What is the probability of getting a number that is less than 6 when tossing a fair die?

Count how many of the outcomes are “favorable” (less than 6). There are five: 1, 2, 3, 4, or 5.
And there are six possible outcomes in total.

Therefore, the probability is $\dfrac{\text{number of favorable outcomes}}{\text{number of possible outcomes}} = \dfrac{5}{6}$.

In math notation we write “P” for probability and put the event in parentheses: **P(less than 6) = 5/6.**

Example 2. On this spinner the number of possible outcomes is eight, because the arrow is equally likely to land on any of the eight wedges. What is the probability of spinning yellow?



There are TWO favorable outcomes (yellow areas) out of EIGHT possible outcomes.

P(yellow) = 2/8 = 1/4.

(Because green and yellow each have two wedges, there are only six possible colors that can result. When we list the possible outcomes, we list the six colors. However, when we figure the probabilities, we must use the eight equal-sized wedges to find the probability.)

By convention, the probability of an event is always at least 0 and at most 1. In symbols: $0 \le \text{P(event)} \le 1$.

A probability of 0 means that the event does not occur; it is impossible. Probability of 1 means that the event is sure to occur; it is certain. A probability near 1 (such as 0.85) means that the event is likely to occur. A probability of 1/2 means that an event is neither likely nor unlikely.

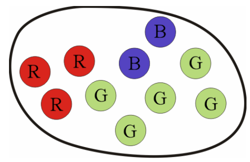
Example 3. What is the probability of rolling 8 on a standard six-sided die?

This is an impossible event, so its probability is zero: P(8) = 0.

Example 4. What is the probability of rolling a whole number on a die?

This is a sure event, so its probability is one. P(whole number) = 1.

1. There are three red marbles, two dark blue marbles, and five light green marbles in Michelle's bag. List all the possible outcomes if you choose one marble randomly from her bag.



2. Michelle chooses one marble at random from her bag. What is the probability that...

   a. the marble is blue?

   b. the marble is not red?

   c. the marble is neither blue nor green?

3. Make up an event with a probability of zero in this situation.

4. Suppose you choose one letter randomly from the word "PROBABILITY."

   a. List all the possible outcomes for this event.

   Now find the probabilities of these events:

   b. P(B)

   c. P(A or I)

   d. P(vowel)

   e. Make up an event for this situation that is likely to occur, yet not a sure event, and calculate its probability.

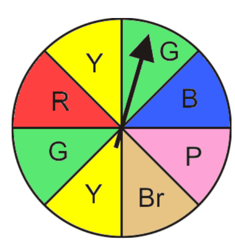
The complement of an event and the probability of "not"

The ***complement*** of any event A is the event that A does *not* occur.

If the probability of event A is a, then the probability of A not happening is simply $1 - a$.

5. The weatherman says that the chance of rain for tomorrow is 1/10. What is the probability of it not raining?

6. The spinner is spun once. Find the probabilities as simplified fractions.



   a. P(green) **b.** P(not green)

   c. P(not pink) **d.** P(not black)

   e. Make up an event for this situation that is not likely, yet not impossible either, and calculate its probability.

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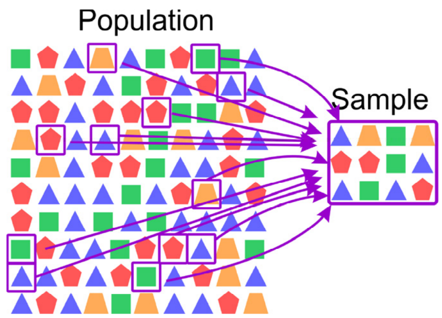
Random Sampling

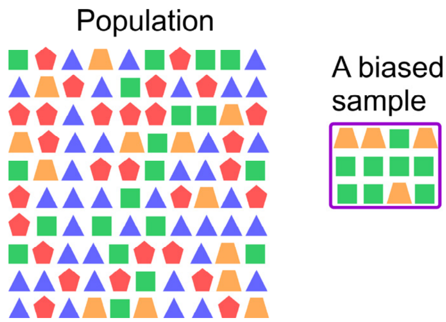
When researchers have a question concerning a large population, they obtain a **sample** (a part) of that population. That is because it is typically impossible to study the entire population.

For example, if you want to know how the citizens of France feel about climate change, you cannot just go and ask every person in France about it. You would choose for example 600 French citizens as your sample and ask them your question.



The way a sample is chosen is very important. Some methods of sampling may produce a sample that is *not* representative of the entire population. We call that a **biased sample**.

For example, if you are studying a student population of 630 in a school with close to an equal number of boys and girls, and you happen to choose a sample of 20 boys, then your sample is biased. It doesn't represent the entire population well.



We need to use **unbiased sampling methods** in order to get a sample that truly represents the population being studied. The best way to avoid biased samples is to select a **random sample.**

The main characteristics of a random sample are:

1. **Randomness:** each member of the population has an equal chance of being selected.

   Let's say a researcher is studying the types of cars Americans own. He decides to interview only people he finds at a local mall because that mall is close to where he lives, so it is convenient for him. His sample is biased because not every member of the US population even has a chance to be selected in his sample. Maybe the people at his local mall are predominantly rich people who own several cars per family, so in that respect those people would not be a good representation of the entire population of the US.

   We call this type of sample a **convenience sample** because it is convenient or easy to obtain.

2. **External selection:** respondents must be chosen by the researcher, not self-selected.

   If our researcher mails a questionnaire to various people across the US asking them to fill it out and return it, his sample is a **voluntary response sample**, which is a biased sample. Some people volunteer to return the questionnaire, but others don't. The people themselves decide whether or not to be a part of the sample.

   Why might this be a problem? Some of the people who would choose to take part may have an external reason to do so. They might want to show off how "good" they are in the particular aspect being studied, or they might just like to speak out about their opinions.

Our researcher could get a true random sample by choosing people randomly from a list of people living in the US and calling them. That way, each person has an equal chance of being selected in the sample (it is random), and the people cannot self-select to take part (the researcher chooses who takes part).

An unbiased sampling method is more likely to produce a representative sample.

1. You are studying whether students in a large college prefer to drink coffee black, with milk, with cream, or with sweetener, or whether they prefer not to drink coffee at all.

   a. Which of the six sampling methods listed below produce a voluntary response sample?

   b. Which methods don't give each member of the student population an equal chance to be selected for the sample?

   c. Which method is likely to produce a sample with only coffee drinkers, overlooking those who don't drink coffee?

   d. Which method will be the most likely to give you a representative (unbiased) sample?

   <u>Sampling Methods</u>

   (1) You interview 80 students in a cafe on the campus.

   (2) You interview 80 students who come in at the main door of the campus.

   (3) You interview the first 80 students you happen to meet on a certain day.

   (4) You choose 80 names randomly from a list of all the students. You call them to interview them.

   (5) You send an email to all the students in the college, asking them to fill in a form on a web page you have set up. You hope to get at least 80 responses.

   (6) You choose 80 names randomly from a list of all the students. You send them an email, asking them to fill in a form on a web page you have set up.

2. A recipe website posts a poll on their home page that any visitor to that website can take. In it, they ask if people are looking for a recipe for a dessert, a main dish, a side dish, bread, or salad. During the course of one Sunday, 4,600 people visit the page, and 252 of them fill in the poll. Explain why the poll results will be based on a biased sample.

Some common random sampling methods are:

1. **Simple random sampling.** Each individual in the sample is chosen randomly and entirely by chance, perhaps by using dice, through pulling names out of a hat, or with a random number generator.

2. **Systematic random sampling.** The individuals of the population are placed in some order, and then each individual at a certain specified interval is selected for the sample.

   For example, a supermarket might study the shopping habits of its customers by choosing every 15th customer who enters the store for the sample.

3. **Stratified random sampling.** The population is first divided into categories (strata) and then a random sample is obtained from each category.

   For example, to study how much sleep students in a particular school get, you might first divide the students into groups by grade levels (the stratification), then select a random sample from each of the grade levels.

3. A population to be studied doesn't have to be of people. A factory produces MP3 players. Out of the 500 units that the factory produces each day, a quality control inspector selects 25 for testing to study their quality and reliability. Which way should he choose those 25 so that his sample would best represent all the MP3 players that the factory produces?

   a. Choose the first 25 produced on a given day.

   b. First choose a number between 1 and 20 randomly. Select the player corresponding to that number, and after that, every 20th player, in the order they were produced that day.

   c. Choose 25 players that have just been finished around 1 PM when the inspector is touring the factory.

   d. Generate 25 random numbers between 1 and 500 and choose the corresponding 25 MP3 players in the order they were produced that day.

4. Ryan has two large fields planted with green beans. He wants to compare the bean plants in one field with the plants in the other. Design a practical sampling method for him to produce an unbiased sample.

Sometimes it is not obvious how a particular sampling method might be biased.

If you are studying students' homework habits in a particular school, it might initially make sense to interview the first 25 students who come into the school in the morning. However, there could be an underlying factor that makes that method biased. What if students who are diligent with their homework also tend to come to school early? In that case, students who are not diligent don't have an equal chance of being selected for your sample. A better method is to use systematic random sampling and to choose, say, every 10th student entering the school for the sample.

5. Heather is studying the effect of how the method of feeding affects the health of a baby during its first year of life. She has already determined that babies who are fed with infant formula get sick more often than babies who are fed with human milk, but she especially wants to find out how often babies who are fed both formula and breast milk get sick. Explain why interviewing mothers in the places below will produce a biased sample:

   a. A pediatrician's office.

   b. A breastfeeding class for new moms.

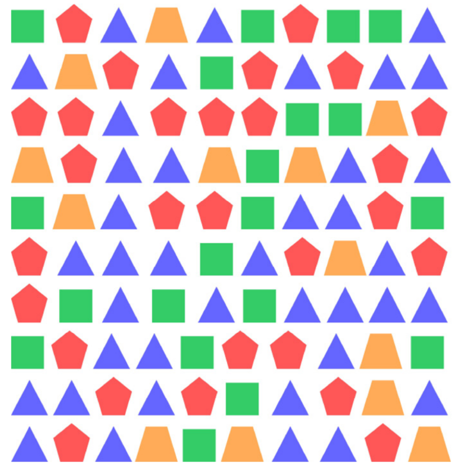
6. **a.** Devise a method that will produce a biased sample based on self-selection, and explain how that would happen, based on Heather's situation in Question 5.

   b. Design a sampling method for Heather that is most likely to produce a representative sample.

7. An organization that helps teenagers with drug problems has set up a telephone hot line for teens to call in to discuss their problems. After a few months of operations, the organization wants to evaluate the effectiveness of their service. Since they don't usually get as many calls on Tuesdays, they decide to choose a particular Tuesday to ask each teen at the end of the call to answer a few questions about how the service has helped. Is this a good method for selecting a sample? Explain.

Using Random Sampling

1. In this activity, you will make several samples of 10 from this population of shapes:



Since the shapes are in a 10 by 10 grid arrangement, you can easily assign a number from 1 to 100 for each shape. To obtain a random sample, you can use one of these ideas or come up with your own.

- Choose a random number between 1 and 10. Then, starting from that number, choose every 10th shape.
- Go to **https://www.random.org/integers/** and generate 10 random integers between 1 and 100. (If the set of numbers contains a duplicate, discard that set and make another.)

Here is an example sample (Sample 1) to help you get started:

It is based on generating these random numbers at the website above: `76 17 51 63 88 29 95 73 40 69`

Generate at least five more samples. Count the number of each kind of shape in each of your samples, and fill in the table. Lastly, calculate the average number of triangles, the average number of squares, and so on.

	Sample 1	**Sample 2**	**Sample 3**	**Sample 4**	**Sample 5**	**Sample 6**	**Averages**
triangles	5						
squares	1						
pentagons	3						
trapezoids	1						
Total	10	10	10	10	10	10	—

2. Now imagine that you haven’t seen the entire population of shapes but you try to infer (conclude) something about the entire population of shapes based on these six samples.

a. Which shape seems to be the most common?

b. Which shape seems to be the least common?

c. List the shapes in order, from the least common to the most common:

d. Based on the average number of the shapes in the six samples, *estimate* how many of each shape there are in the entire population of 100 shapes:

_____ triangles _____ squares _____ pentagons _____ trapezoids

Here are some important points to realize and remember concerning random sampling. You probably noticed these facts while doing the activity:

1. **Random samples vary.** The differences occur simply because their members are indeed chosen from the population at random.
2. **A random sample is more likely to be unbiased, but that is not 100% certain.** In other words, it is possible for a random sample *not* to be representative of the population as a whole. However, the chances of a random sample being unbiased are greater than the chances of a non-random sample being unbiased.
3. **It is better to base inferences about the population on more than one sample.** However, if you cannot obtain more than one sample, then it is recommended to increase the sample size as much as possible, because a large sample represents the whole population better than a small one.

Example 1. Three people are running for mayor in a town with 20,000 voters. Two companies conducted separate polls of 350 people, asking who they would vote for in the final election. Here are the results:

	Smith	Harrison	Jones
Poll 1	63	220	67
Poll 2	53	238	59

Based on these results, what can we conclude about the results of the final election?

In both polls, Harrison is winning and by a large margin. In other words, far more people are claiming that they will vote for Harrison than for Smith or Jones. So we can fairly confidently conclude that <u>Harrison will be the winner of the actual election</u>.

Not only that, but in both polls Jones did better than Smith. So <u>it is likely, but not sure, that Jones will beat Smith in the actual election</u>. We cannot say that for sure because the differences are small: 4 and 6 votes.

We can also quantify the election results (use actual numbers). In Poll 1, 220 is more than three times 63 or 67. In Poll 2, 238 is more than four times 53 and about four times 59. Based on those, we can say that <u>Harrison will get roughly 3-4 times more votes than either of his opponents</u>.

Example 2. The data below presents the results of three different samples from a study about how students prefer to drink coffee.

	Black	With milk	With cream	Milk and sugar	Cream and sugar	Totals
Sample 1	12	21	24	36	37	130
Sample 2	9	23	22	37	39	130
Sample 3	14	18	20	36	42	130

What can we infer based on the data?

(1) Looking at the numbers carefully we can see that in each of the samples “Cream and sugar” was the winner and “Milk and sugar” came fairly close behind it.

(2) The two options “With milk” and “With cream” are also close to each other, but we cannot say for sure which of them is preferred, because in Samples 1 and 3, “With cream” beats “With milk”, whereas in sample 2 it is the opposite way.

(3) Drinking black coffee is the least popular option in all three samples.

(4) We can quantify the results. For example, “Cream and sugar” is the favorite of roughly 40/130 = 4/13 of the students, and 4/13 is almost 4/12 = 1/3. So we can state that <u>almost 1/3 of the students prefer to drink their coffee with cream and sugar</u>. You can make similar statements using approximate fractions for the other options.

What kinds of inferences can you make about the entire population based on random samples?

Based on what the data demonstrates, you may be able to…

- state which option is the most or least, the best or worst, the winner or loser, *etc*.
- compare two options as better or worse, more or less, *etc*.
- quantify the above statements with numbers, fractions, or percentages:
  How much more or less is one option than another?
- find trends: identify an increase or a decrease in some quantity as some other quantity, such as time, increases or decreases.

3. A large workplace conducted a survey of their employees' sleeping hours. They took two samples of 65 people, one week apart. What can you infer based on these results?

	< 5 h	5 h	6 h	7 h	8 h	> 8h	Totals
Sample 1	1	4	21	32	6	1	65
Sample 2	2	8	23	26	4	2	65

4. A music band wanted to find out which of their songs their audience likes best. They randomly chose some people to be interviewed after two of their concerts, asking them what their favorite song was. The results are in the table at the right.

   What conclusions can you draw from the data?

Songs	Concert 1 (Sample 1)	Concert 2 (Sample 2)
"Love You"	4	3
"My Best"	9	11
"Never Again"	7	5
"Sunshine"	5	6
Totals	25	25