Solving Systems of Equations by Substitution

While graphing is a valid way to solve systems of equations, it is not the best since the coordinates of the intersection point may be decimal numbers, and even irrational. In this lesson you will learn one algebraic method for solving systems of equations, called **the substitution method**.

Example 1. Note that the second equation in this system of equations is of the form "y = something", and this "something" only involves the variable x.

$$\begin{cases} 5x - 2y = 16 \\ y = -2x + 1 \end{cases}$$

This means we can replace y in the *first* equation by the expression that y equals (which is $-2x + 1$), and the result will be an equation that will have **only one unknown**, x.

We can then solve that equation for x. Once we know the value of x, we can substitute that value in one of the original equations, and thus get an equation only in y. Solving that, we will find the value of y.

So, let's do all that!

Since $y = -2x + 1$, we will substitute $-2x + 1$ in place of $\boldsymbol{y}$ in the first equation. →

$$\begin{aligned} 5x - 2\boldsymbol{y} &= 16 \\ 5x - 2(-2x + 1) &= 16 \\ 5x + 4x - 2 &= 16 \\ 9x - 2 &= 16 \\ 9x &= 18 \\ x &= 2 \end{aligned}$$

Once we solve that $x = 2$, we then substitute **2** in place of x in the second equation:

$$y = -2x + 1 = -2(\mathbf{2}) + 1 = -3$$

So, the solution is $x = 2$ and $y = -3$. Let's check:

$$\begin{cases} 5(2) - 2(-3) \stackrel{?}{=} 16 \\ -3 \stackrel{?}{=} -2(2) + 1 \end{cases} \rightarrow \begin{cases} 10 + 6 = 16 \ ✓ \\ -3 = -4 + 1 \ ✓ \end{cases}$$

1. Solve each system of equations using the substitution method. Check your solutions.

a. $\begin{cases} x + y = -9 \\ y = 3x - 1 \end{cases}$

b. $\begin{cases} x = 3y - 11 \\ 2x + 2y = 10 \end{cases}$

Example 2. Here, we have added line numbers to the equations, just to be able to reference each equation.

$$\begin{cases} 6x - 2y = -12 & (1) \\ x + 3y = -10 & (2) \end{cases}$$

This time, neither equation is in the form "y = something" or "x = something", so we cannot substitute any expression directly to either equation. But, we can solve either x or y from either equation, and then use the substitution method.

Let's solve x from equation 2, which is relatively simple to do. We mark (2) in front of it to signify it is equation 2 we are using.

$$\begin{aligned} (2)\quad x + 3y &= -10 \\ x &= -3y - 10 \end{aligned}$$

Now, we substitute the expression $\mathbf{-3y - 10}$ in place of x in equation 1, resulting in an equation that only has one unknown.

Then we solve this equation for y.

$$\begin{aligned} (1)\quad 6x - 2y &= -12 \\ 6(\mathbf{-3y - 10}) - 2y &= -12 \\ -18y - 60 - 2y &= -12 \\ -20y - 60 &= -12 && \mid + 60 \\ -20y &= 48 && \mid \div(-20) \\ y &= -48/20 = -12/5 \end{aligned}$$

Then it's time to substitute this value −12/5 in place of y in *either* of the original equations, to find the value of x. Let's use equation 2, since in it, x does not have any coefficients so the calculation will be shorter.

$$\begin{aligned} (2)\quad x + 3y &= -10 \\ x &= -3y - 10 \\ x &= -3(-12/5) - 10 \\ x &= 36/5 - 10 = 36/5 - 50/5 = -14/5 \end{aligned}$$

The solution is (−14/5, −12/5).

2. Solve each system of equations using the substitution method. Check your solutions (always!).

a. $\begin{cases} x - 4y = -2 \\ y = 5 - 2x \end{cases}$	**b.** $\begin{cases} x = 10y + 1 \\ (1/2)x - y = 3 \end{cases}$	**c.** $\begin{cases} 3x = 3(y - 1) \\ y = 5x \end{cases}$

3. Find the errors that the students made in solving these system of equations.

a.

$$\begin{cases} x + 3y = -5 & \textbf{(1)} \\ 2x + y = 3y - 1 & \textbf{(2)} \end{cases}$$

↓

$$\begin{cases} x = -3y - 5 & \textbf{(1)} \\ 2x + y = 3y - 1 & \textbf{(2)} \end{cases}$$

↓

$$\begin{aligned} \textbf{(1)} \quad \mathbf{(-3y - 5)} + 3y &= -5 \\ -3y - 5 + 3y &= -5 \\ -5 &= -5 \end{aligned}$$

What now? Does this mean y can be any number?

b.

$$\begin{cases} -4x - y = 2(x - 3) & \textbf{(1)} \\ x + y = 3 & \textbf{(2)} \end{cases}$$

↓

$$\begin{cases} -4x - y = 2(x - 3) & \textbf{(1)} \\ y = 3 - x & \textbf{(2)} \end{cases}$$

↓

$$\begin{aligned} \textbf{(1)} \quad -4x - (\mathbf{3 - x}) &= 2(x - 3) \\ -3x - 3 &= 2x - 6 \\ -5x - 3 &= -6 \\ -5x &= -3 \\ x &= 3/5 \end{aligned}$$

The solution is 3/5.

4. Find the error in this solution, also.

$$\begin{cases} 5x - 2y = 10 & \textbf{(1)} \\ 3x - 8y = 4y - 48 & \textbf{(2)} \end{cases}$$

↓

$$\begin{cases} 5x = 2y + 10 & \textbf{(1)} \\ 3x - 8y = 4y - 48 & \textbf{(2)} \end{cases}$$

↓

$$\begin{aligned} \textbf{(2)} \quad 3\mathbf{(2y + 10)} - 8y &= 4y - 48 \\ 6y + 30 - 8y &= 4y - 48 \\ -2y + 30 &= 4y - 48 \\ -6y &= -78 \\ y &= 13 \end{aligned}$$

Substituting $y = 13$ to the first equation:

$$\begin{aligned} \textbf{(1)} \quad 5x - 2(13) &= 10 \\ 5x - 26 &= 10 \\ 5x &= 36 \\ x &= 7.2 \end{aligned}$$

(*Continues in the column on the right.*)

However, (7.2, 13) does NOT fulfill the second equation:

$$\begin{aligned} \textbf{(2)} \quad 3x - 8y &= 4y - 48 \\ 3(7.2) - 8(13) &\stackrel{?}{=} 4(13) - 48 \\ 21.6 - 104 &\neq 4 \end{aligned}$$

What happens if a system of equations has no solutions, or an infinite number of solutions? How will that look like in an algebraic solution?

Example 3. The system below actually has no solutions. Let's see what happens when we try to solve it.

$$\begin{cases} 8x + y = 3(1 + y) & (1) \\ -4x + y = -5 & (2) \end{cases}$$

First we solve y from the second equation:

$$\begin{aligned} (2)\quad -4x + y &= -5 \\ y &= 4x - 5 \end{aligned}$$

Now we substitute that for y in the first equation:

$$\begin{aligned} (1)\quad 8x + (4x - 5) &= 3[1 + (4x - 5)] \\ 12x - 5 &= 3(1 + 4x - 5) \\ 12x - 5 &= 3(4x - 4) \\ 12x - 5 &= 12x - 12 \\ -5 &= -12 \end{aligned}$$

We end up with a false equation. That is how we can see that the system has no solutions.

Example 4. $\begin{cases} 3x - y = 2(x - 3) & (1) \\ x + 5y = 6(y - 1) & (2) \end{cases}$

First we solve y from the first equation:

$$\begin{aligned} (1)\quad 3x - y &= 2(x - 3) \\ -y &= 2(x - 3) - 3x \\ -y &= 2x - 6 - 3x \\ y &= x + 6 \end{aligned}$$

Now we substitute that for y in the second equation:

$$\begin{aligned} (2)\quad x + 5(x + 6) &= 6[(x + 6) - 1] \\ x + 5x + 30 &= 6[x + 6 - 1] \\ 6x + 30 &= 6x + 30 \end{aligned}$$

Right here we can see that any x satisfies the above equation. Or, we could continue by subtracting 30 and $6x$ from both sides and arrive at the identity $0 = 0$.

Arriving at this situation means that the original two equations are equivalent (the two lines are the same line). However, it doesn't mean both x and y can be just anything. The pair (x, y) has to be <u>a point on this line</u>.

The solution needs to include the equation that x and y satisfy (the equation of the line). It can be *either* of the original equations, or an equivalent equation. Here it makes sense to use $y = x + 6$ since it is so simple.

<u>Solution:</u> All points (x, y) that satisfy the equation $y = x + 6$.

5. Solve each system of equations using the substitution method. Use extra paper if necessary.

a. $\begin{cases} 2x = 14 - 2y \\ x + y = 2 \end{cases}$

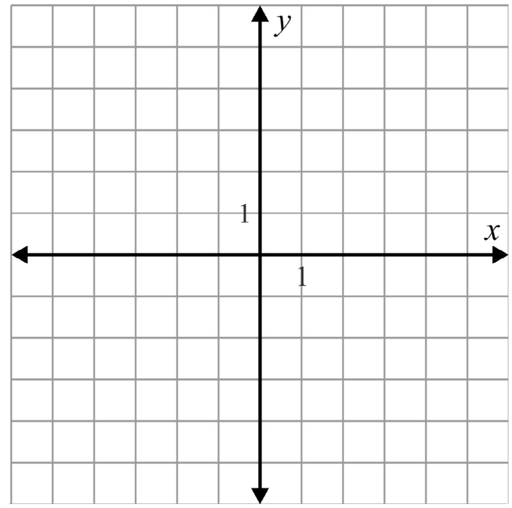
b. $\begin{cases} 2 - 4x = y + 8 \\ x - y = 5x + 6 \end{cases}$

6. Solve each system of equations. Use extra paper if necessary. Check your solutions (always do that!).

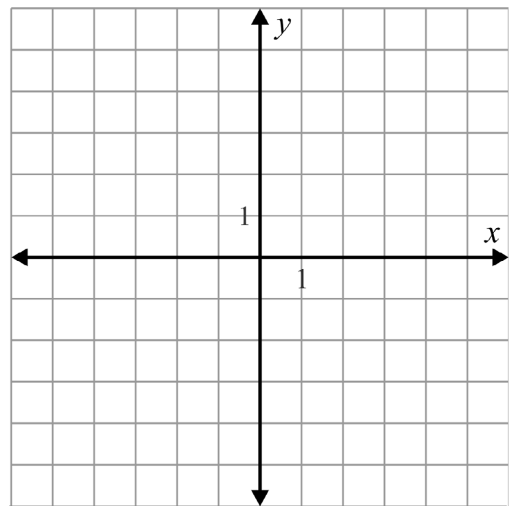
a. $\begin{cases} 4x - 2y = 5 \\ y = 5x + 8 \end{cases}$	**b.** $\begin{cases} 4x - 5y = 12 \\ y = 2(5 - x) \end{cases}$
c. $\begin{cases} x - 2y = 5 - 3.5y \\ 2(5 - x) = 3y \end{cases}$	**d.** $\begin{cases} -4x - 3y = 2 \\ y + 4x = -6 \end{cases}$
e. $\begin{cases} -9x = -2y \\ 9x - 2y = 2 \end{cases}$	**f.** $\begin{cases} 2(2x - y) = 2 \\ x + 6y = 1 \end{cases}$

7. Solve each system of equations by substitution. Then graph the lines. Verify that the intersection point of the lines is the solution you found algebraically.

a. $\begin{cases} 2x - 5y = 20 \\ -x - y = 3 \end{cases}$



b. $\begin{cases} -x + 5y = 20 \\ y = 3x \end{cases}$



8. Solve each system of equations. Round the coordinates to two decimal digits.

a. $\begin{cases} 0.2x - 0.7y = 1 \\ x + y = 4 \end{cases}$

b. $\begin{cases} 1.2x - 3y = -2 \\ y = -0.6x + 5 \end{cases}$

9. These are for additional practice as needed. Use extra paper if necessary, and check your solutions.

a. $\begin{cases} x = -(y-2) \\ 3y = 10(2-x) \end{cases}$	**b.** $\begin{cases} 4x - 3y = -1 \\ 3x + y = -2 \end{cases}$
c. $\begin{cases} x = -3(y+8) \\ -x + 3y = 0 \end{cases}$	**d.** $\begin{cases} 3(x-2) + y = 12 \\ y = 4x - 3 \end{cases}$
e. $\begin{cases} 2x - 8y = 15 \\ -x + y = 20 \end{cases}$	**f.** $\begin{cases} x - 3(3y+2) = 1 \\ 3x + 5y = -1 \end{cases}$